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The Degree Sequence of Random Graphs from Subcritical Classes†

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In this work we determine the expected number of vertices of degree $k = k(n)$ in a graph with n vertices that is drawn uniformly at random from a *subcritical graph class*. Examples of such classes are outerplanar, series-parallel, cactus and clique graphs. Moreover, we provide exponentially small bounds for the probability that the quantities in question deviate from their expected values.

1. Introduction and results

One of the central questions of interest in theoretical computer science is the analysis of algorithms. Here one usually distinguishes between *worst-case analysis* and *average-case analysis*. From a practical point of view, an average-case analysis is particularly important when the worst-case analysis does not result in satisfactory quality characteristics about the given algorithm: it is possible that the algorithm is efficient in real-world scenarios (the 'typical case'), although a bad worst-case behaviour can be mathematically shown. In order to prove qualitatively strong and meaningful results about the average-case behaviour of a particular algorithm, we usually require precise knowledge about properties of 'typical' input instances.

In the context of graph algorithms, an average-case analysis can often easily be achieved if we assume the uniform distribution on the set of all graphs with a given number of vertices: one can then model a 'typical' input by the classical Erdős–Rényi random graph, and thus use the wide and extensive knowledge about random graphs (see the two excellent monographs [5] and [15]), to derive properties that can be used to analyse performance measures such as running time or approximation ratios.

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The picture changes dramatically if we are interested in *natural* graph classes. A standard example that has evolved over the last decade as a reference model in this context is the class of *planar* graphs. The random planar graph R_n was first investigated in [7] by Denise, Vasconcellos and Welsh and has attracted considerable attention since then. We mention selectively a few results. McDiarmid, Steger and Welsh [16] showed the surprising fact that R_n does not share the 0–1 law known from standard random graph theory: the probability of connectedness is bounded away from 0 and 1 by positive constant values; moreover, the situation is similar if the average degree is fixed [13]. These results relied on a (crude) counting of the number of planar graphs with n vertices. A breakthrough occurred with the recent results of Giménez and Noy [14], who not only managed to determine the asymptotic value of the number of planar graphs with n vertices, but also showed that the number of edges in R_n is asymptotically normally distributed. Moreover, they studied the number of connected and 2-connected components in R_n. The proofs of these results are based on singularity analysis of generating functions, a powerful method from analytic combinatorics that has led to many beautiful results: see the book by Flajolet and Sedgewick [10].

Our results. In this paper we further elaborate and extend significantly an approach that was used in [3] to obtain the degree sequence and subgraph counts of random dissections of convex polygons. More precisely, we exploit the so-called Boltzmann sampler framework by Duchon, Flajolet, Louchard and Schaeffer [9] to reduce the study of degree sequences to properties of sequences of *independent* and *identically distributed* random variables. Hence, we can – and do – use many tools developed in classical random graph theory to obtain extremely tight results.

Our first main contribution is a general *framework* that allows us to derive mechanically the degree distribution of random graphs from certain 'nice' graph classes, which are 'subcritical' in a well-defined analytic sense: see Section 3 for details. Our framework can be readily applied to obtain the degree sequence of random graphs from 'simple' classes, such as Cayley trees – *i.e.*, (non-plane) labelled trees – or graphs which have the property that their maximal 2-connected components (or equivalently, blocks) have a simple structure. We mention as examples cactus graphs, where the blocks are cycles, and clique graphs, where the blocks are complete graphs. The main contribution of our work consists of two involved applications of this framework.

A graph is called *series-parallel* (SP) if it does not contain a subdivision of the complete graph K_4, or equivalently if it does not contain K_4 as a minor. Hence, the class of SP graphs is a subclass of all planar graphs. Moreover, an *outerplanar* graph is a planar graph that can be embedded in the plane so that all vertices are incident to the outer face. Outerplanar graphs are characterized as those graphs that do not contain a K_4 or a $K_{2,3}$ minor. The classes of outerplanar and SP graphs are often used as the first non-trivial test cases for results about the class of all planar graphs.

As two important applications of our framework we derive the degree distribution of random outerplanar and random SP graphs, and show that the number of vertices of degree $k = k(n)$ (where k is not allowed to grow too fast) is concentrated around a specific value with very high probability. In particular, we show the following result, where we write '$\mathsf{deg}(k; G)$' for the number of vertices of degree k in G, and '$(1 \pm \alpha)X$' for the interval $((1-\alpha)X, (1+\alpha)X)$.

Theorem 1.1. *Let O_n be a graph drawn uniformly at random from the set of all labelled connected outerplanar graphs with n vertices. There are constants $C_{OP}, c > 0$ and a function $op : \mathbb{N} \to \mathbb{R}$ such that, for any $0 < \varepsilon, \delta < 1$, the following is true. Let $1 \leqslant k \leqslant (C_{OP} - \delta)\log n$. Then, for sufficiently large n,*

$$\mathbb{P}\big[\mathsf{deg}(k; \mathsf{O}_n) \in (1 \pm \varepsilon)\, op(k)n\big] \geqslant 1 - e^{-c\frac{\varepsilon^2}{(\log(\varepsilon^{-1})+k)^2}\frac{op(k)n}{k}}.$$

The same is true for random SP graphs, for a suitably chosen function $sp : \mathbb{N} \to \mathbb{R}$ and a constant $C_{SP} > 0$.

We want to remark at this point that we determine explicitly the functions op and sp in the above theorem, and refer the reader to Sections 5 and 6 for a more precise formulation of the results. Moreover, by deriving precise asymptotics for the behaviour when $k \to \infty$, we give strong evidence that the constants C_{OP} and C_{SP} are best possible. In other words, we conjecture that the maximum degree of a random outerplanar graph is $\sim C_{OP} \log n$ and that the maximum degree of a random SP graph is $\sim C_{SP} \log n$.

The number of vertices of a given degree in random outerplanar and series-parallel graphs is also studied by Drmota, Giménez and Noy [8], independently from our work. Using different techniques, the authors show for *fixed* k that the number of vertices of degree k is asymptotically normally distributed, with expectation and variance linear in n.

Techniques. All graph classes considered in this paper allow a so-called *decomposition*, which is a *description* of the class in terms of general-purpose combinatorial constructions. These constructions appear frequently in modern systematic approaches to asymptotic enumeration and random sampling of combinatorial structures. It is beyond the scope of this work to survey these results, and we refer the reader to [10] and references therein for a detailed exposition.

One benefit of the knowledge of the decomposition is that it allows us to develop *mechanically* algorithms that sample objects from the graph class in question by using the framework of *Boltzmann samplers*. This framework was introduced by Duchon, Flajolet, Louchard and Schaeffer in [9], and was extended by Fusy [12] to obtain an (expected) linear-time approximate-size sampler for planar graphs. Here we just present the basic ideas of this framework. Let $\mathcal{G}$ be a class of labelled graphs, and let $|\gamma|$ denote the number of labelled vertices in any $\gamma \in \mathcal{G}$. In the Boltzmann model of parameter x, we assign to any object $\gamma \in \mathcal{G}$ the probability

$$\mathbb{P}_x[\gamma] = \frac{1}{G(x)} \frac{x^{|\gamma|}}{|\gamma|!}, \tag{1.1}$$

if the expression above is well defined, where $G(x)$ is the exponential generating function enumerating the elements of $\mathcal{G}$. It is easy to see that the expected size of an object in $\mathcal{G}$ under this probability distribution is $\frac{xG'(x)}{G(x)}$. A *Boltzmann sampler* $\Gamma G(x)$ for $\mathcal{G}$ is an algorithm that generates graphs from $\mathcal{G}$ according to (1.1). In [9, 12] several general procedures which translate common combinatorial construction rules such as union, set, *etc*., into Boltzmann samplers are given. Notice that the probability above only depends on the choice of x and on the size of γ, so every object of the same size has the *same* probability of being generated. This means that if we condition on the output being of a certain size n, then the Boltzmann sampler $\Gamma G(x)$ is a *uniform* sampler of the class $\mathcal{G}_n$.

Outline. In Section 3 we introduce the notion of a 'nice' class, and relate it to the concept of analytic subcriticality. By exploiting the Boltzmann sampling framework we then develop a generic algorithm that samples uniformly at random graphs with n vertices from any nice class of graphs. The main result of Section 3 is Theorem 3.3, which gives an almost complete picture of the degree distribution of such random graphs, and is hence at the heart of our work. Section 4 presents some sample applications of Theorem 3.3, and provides some tools that considerably simplify its utilization. In Section 5 we use this result to derive the degree distribution of a random outerplanar graph. Finally, in Section 6 we first present a new method to derive the degree distribution of random 2-connected series-parallel (SP) graphs and then apply Theorem 3.3 in order to obtain the degree distribution of a random SP graph.

2. Preliminaries

In our proofs we will often need to bound the probability that certain random variables assume values far away from their expectation. The following two lemmas will be very helpful for the case where the variables are binomial or Poisson-distributed; the presentation is as in [15].

Lemma 2.1 (Chernoff bounds). *Let X be a binomially distributed random variable and let $t > 0$. Then*

$$\mathbb{P}[|X - \mathbb{E}[X]| > t] \leqslant 2\exp\left(-\frac{t^2}{2(\mathbb{E}[X] + t/3)}\right). \tag{2.1}$$

Lemma 2.2. *Let X be distributed like a Poisson variable with mean $\mu > 1$. There exists a constant $C > 0$ such that, for every $0 < \varepsilon < 1$,*

$$\mathbb{P}[|X - \mu| \geqslant \varepsilon\mu] \geqslant 1 - e^{-C\varepsilon^2\mu}.$$

A more general tool that we shall apply several times is Talagrand's inequality: see the book by Janson, Łuczak and Ruciński [15] for a detailed introduction. Intuitively, it provides strong bounds for the probability that a function defined on a set of independent random variables deviates significantly from its expectation, when the value of the function is not affected much by small changes in each one of its arguments.

Theorem 2.3 (Talagrand's inequality). *Let $Z_1, \dots, Z_N$ be independent random variables taking values in the sets $\Lambda_1, \dots, \Lambda_N$ respectively. Let $\Lambda = \Lambda_1 \times \cdots \times \Lambda_N$. Let $f : \Lambda \to \mathbb{R}$ be* *a function and set* $X = f(Z_1, \dots, Z_N)$. *Assume that there are quantities* c_k, $k = 1, \dots, N$ *satisfying the following.*

(a) *If* $z, z' \in \Lambda$ *differ only in the kth coordinate, then* $|f(z) - f(z')| \leqslant c_k$.

(b) *There is an increasing function* ψ *satisfying the following. Let* $z \in \Lambda$ *and* $r \in \mathbb{R}$ *such that* $f(z) \geqslant r$. *Then there exists a set* $J \subseteq \{1, \dots, N\}$ *with* $\sum_{i \in J} c_i^2 \leqslant \psi(r)$, *such that, for all* $y \in \Lambda$ *with* $y_i = z_i$ *when* $i \in J$, *we have* $f(y) \geqslant r$.

Then, if $\mathbb{M}[X]$ *denotes the median of* X, *for every* $t \geqslant 0$ *we have*

$$\mathbb{P}[|X - \mathbb{M}[X]| \geqslant t] \leqslant 4 \exp\left(-\frac{t^2}{4\psi(\mathbb{M}[X] + t)}\right). \tag{2.2}$$

The following technical lemma is needed in the proof of Theorem 5.1. Its proof uses the well-known saddle point method for complex integrals and can be found in the Appendix.

Lemma 2.4. *Let* α, β, γ *be constants such that* $\alpha + \beta \geqslant 0$. *For large* n,

$$[z^n]\left(e^{\frac{\alpha z + \beta z^2}{1-z}} \cdot \frac{1}{(1-z)^\gamma}\right) = (1 + o(1)) e^{2\sqrt{(\alpha+\beta)n}} \cdot n^{\frac{\gamma}{2} - \frac{3}{4}} \cdot e^{-\frac{\alpha}{2} - \frac{3\beta}{2}} (\alpha + \beta)^{-\frac{\gamma}{2} + \frac{1}{4}} \pi^{-\frac{1}{2}} 2^{-1}. \tag{2.3}$$

Notation. Let us introduce some notation that will be used extensively in the following sections. Let $\mathcal{G}$ be a class of labelled graphs. We denote by $\mathcal{G}_n$ the subset of graphs in $\mathcal{G}$ that have precisely n labelled vertices, and assume without loss of generality that the labels are from $\{1, \dots, n\}$. We set $g_n := |\mathcal{G}_n|$. Moreover, we write $G(x) = \sum_{n \geqslant 0} g_n \frac{x^n}{n!}$ for the corresponding *exponential generating function* (egf).

In the following we will frequently use the *pointing* and *derivative* operators. Given a class $\mathcal{G}$, we define $\mathcal{G}^\bullet$ as the class of *pointed* (or *rooted*) graphs, where a vertex is distinguished from all other vertices. The *derived* class $\mathcal{G}'_{n-1}$ is obtained by removing the label n from every object in $\mathcal{G}_n$, such that the obtained objects have $n - 1$ labelled vertices, *i.e.*, vertex n can be considered as a distinguished vertex that does not contribute to the size. Consequently, there is a bijection between the classes $\mathcal{G}'_{n-1}$ and $\mathcal{G}_n$. We set $\mathcal{G}' := \bigcup_{n \geqslant 0} \mathcal{G}'_n$. On a generating function level, the pointing operation corresponds to taking the derivative with respect to x, and multiplying it by x, that is, $G^\bullet(x) = xG'(x)$. Similarly, the egf of $\mathcal{G}'$ is simply $G'(x)$. Finally, we denote by $\rho_{\mathcal{G}}$ the dominant singularity of G, which will be in all considered cases unique, and we write $|\mathsf{G}|$ for the number of labelled vertices in $\mathsf{G} \in \mathcal{G}$.

3. A framework for nice graph classes

The aim of this section is to develop a general framework that will allow us mechanically to give tight bounds for the number of vertices of degree k in a random graph drawn from a graph class that satisfies certain technical assumptions. Before we state our main result formally, let us introduce some notation. We denote by $\mathcal{Z}$ the graph class consisting of one single labelled vertex. Furthermore, for two graph classes $\mathcal{X}$ and $\mathcal{Y}$, we denote by $\mathcal{A} = \mathcal{X} \times \mathcal{Y}$ the *Cartesian product* of $\mathcal{X}$ and $\mathcal{Y}$ followed by a relabelling step, so as to guarantee that all labels are distinct. Note that the relation '$\mathcal{A} = \mathcal{X} \times \mathcal{Y}$' expresses the fact that there is a bijection between the elements of $\mathcal{A}$ and pairs of elements from $\mathcal{X}$ and $\mathcal{Y}$, but it does not provide any information about what this bijection looks like, *i.e.*, how to construct a graph in $\mathcal{A}$ from two graphs in $\mathcal{X}$ and $\mathcal{Y}$. The same is true for the operators described below. We denote by $\mathrm{SET}(\mathcal{X})$ the graph class such that each object in it is an unordered collection of graphs in $\mathcal{X}$. Finally, the class $\mathcal{X} \circ \mathcal{Y}$ consists of all graphs that are obtained from graphs from $\mathcal{X}$, where each vertex is replaced by a graph from $\mathcal{Y}$. Here we will usually assume that $\mathcal{Y}$ is a class of rooted graphs, which simply means that we attach a graph G_v from $\mathcal{Y}$ at each vertex v by identifying the root of G_v with v. This set of combinatorial operators (Cartesian product, set, and substitution) appears frequently in modern theories of combinatorial analysis [10, 2, 18] as well as in systematic approaches to random generation of combinatorial objects [9, 11]. For a very detailed description of these operators and numerous applications, we refer to [10].

With this notation we may now define the graph classes we are going to consider. Here we say that a graph is *biconnected* if is either 2-connected, or isomorphic to a single edge.

Definition 1. Let $\mathcal{G}$ be a class of labelled graphs, and let $\mathcal{B} = \mathcal{B}(\mathcal{G}) \subset \mathcal{G}$ be the subclass of biconnected graphs in $\mathcal{G}$. We say that $\mathcal{G}$ is *nice* if it fulfils the following two conditions.

(i) $\mathcal{G}^\bullet$ satisfies

$$\mathcal{G}^\bullet = \mathcal{Z} \times \mathrm{SET}(\mathcal{B}' \circ \mathcal{G}^\bullet). \tag{3.1}$$

(ii) The egf $B(x)$ enumerating $\mathcal{B}$ has a unique singularity at $\rho_{\mathcal{B}}$ and satisfies $\rho_{\mathcal{B}} B''(\rho_{\mathcal{B}}) > 1$.

This definition states that nice classes allow the following decomposition: a rooted graph is a collection of rooted *bi*connected graphs, which are 'glued' together at their roots, and every non-root vertex in them is again substituted by some graph from the class. Note that all graphs from a nice class have the property that all their blocks are contained in $\mathcal{B}$. Moreover, if $\mathcal{B}$ is any class of biconnected graphs, and $\mathcal{G}$ is the class of all connected graphs all of whose blocks (*i.e.*, maximal biconnected subgraphs) are in $\mathcal{B}$, then $\mathcal{G}^\bullet$ satisfies (3.1).

Probably the most prominent examples that fit into this framework are classes with forbidden biconnected minors, such as (connected) planar, outerplanar, and series-parallel graphs, or cactus, block graphs and many kinds of trees (like Cayley trees). On the other hand, the second condition in the above definition is more restrictive. In particular, it says that the composition schema described in (3.1) is *subcritical*, thus imposing heavy restrictions on the analytic behaviour of $G^\bullet(x)$. As we shall see later, planar graphs are not nice in the above sense, but, for example, outerplanar graphs and series-parallel graphs are.

The following statement gives us precise asymptotic information about the number of graphs in a nice class. The proof is straightforward.

Lemma 3.1. *Suppose that the* egf $G^\bullet(x)$ *of a nice class* $\mathcal{G}^\bullet$ *is aperiodic.*[1] *Then* $G^\bullet(x)$ *has a unique finite singularity* $\rho_{\mathcal{G}}$ *and there exists a* $c > 0$ *such that* $g_n^\bullet \sim cn^{-3/2} \cdot \rho_{\mathcal{G}}^{-n} \cdot n!$. *Moreover,* $G^\bullet(\rho_{\mathcal{G}}) < \rho_{\mathcal{B}}$.

[1] A function $f(z)$ is called aperiodic if there is no $h(z)$ such that $f(z) = h(z^d)$, where $d \in \{2, 3, \dots\}$.

Proof. It is easily verified that $G^\bullet(x)$ belongs to the *implicit function schema*, as it is defined in [10, p. 467]. In particular, condition (ii) in Definition 1 and the positivity of the coefficients of $B(x)$ guarantee a solution to the characteristic system

$$r \cdot e^{B'(s)} = s, \quad r \cdot B''(s) \cdot e^{B'(s)} = 1,$$

where $r = \rho_{\mathcal{G}}$ and $s = G^\bullet(\rho_{\mathcal{G}})$. In particular, $s < \rho_{\mathcal{B}}$, as $\rho_{\mathcal{B}} B''(\rho_{\mathcal{B}}) > 1$. The result follows by applying Theorem VII.3 from [10]. □

The remainder of this section is structured as follows. In the next subsection we shall define an algorithm that generates graphs from a nice class $\mathcal{G}$ according to the Boltzmann model for $\mathcal{G}$. This sampler will provide us with the necessary intuition about how the number of vertices in a random graph from $\mathcal{G}_n$ evolves during its generation. Then, in Section 3.2 we exploit this sampler to prove our main result (Theorem 3.3) for nice classes.

3.1. A sampler for nice graph classes

Recall that due to (3.1) a rooted graph from a nice class $\mathcal{G}^\bullet$ of graphs can be viewed as a set of rooted biconnected graphs, which are 'glued' together at their roots, and every vertex in them is substituted by a rooted connected graph. A sampler for $\mathcal{G}^\bullet$ reverses this description: it starts with a single vertex, attaches to it a random set of biconnected graphs, and proceeds recursively to substitute every newly generated vertex by a rooted connected graph.

Let us now define formally the generic sampler. For this we need some additional notation. Let $\rho_{\mathcal{G}}$ and $\rho_{\mathcal{B}}$ be the singularities of the egfs enumerating $\mathcal{G}$ and $\mathcal{B}$. Define

$$\lambda_{\mathcal{G}} := B'(G^\bullet(\rho_{\mathcal{G}})),$$

and let $\Gamma B'(x)$ be a Boltzmann sampler for $\mathcal{B}'$, *i.e.*, $\Gamma B'(x)$ samples according to the Boltzmann distribution (1.1) with parameter x for $\mathcal{B}'$. Note that $\lambda_{\mathcal{G}} < \infty$, as, due to Lemma 3.1, $G^\bullet(\rho_{\mathcal{G}}) < \rho_{\mathcal{B}}$. The sampler $\Gamma G^\bullet$ for $\mathcal{G}^\bullet$ is defined recursively as follows:

$\Gamma G^\bullet$: $\gamma \leftarrow$ a single node r
  $k \leftarrow \mathrm{Po}(\lambda_{\mathcal{G}})$ (⋆)
  for $(j = 1, \dots, k)$
    $\gamma' \leftarrow \Gamma B'(G^\bullet(\rho_{\mathcal{G}}))$, discard the labels of γ' (⋆⋆)
    $\gamma \leftarrow$ merge γ and γ' at their roots
  foreach vertex $v \neq r$ of γ
    $\gamma_v \leftarrow \Gamma G^\bullet$, discard the labels of γ_v (∗)
  replace all nodes $v \neq r$ of γ with γ_v
  label the vertices of γ uniformly at random
  return γ

The following lemma is an immediate consequence of the compilation rules in [9, 12].

Lemma 3.2. *Let $\gamma \in \mathcal{G}^\bullet$. Then*

$$\mathbb{P}[\Gamma G^\bullet = \gamma] = \frac{\rho_{\mathcal{G}}^{|\gamma|}}{|\gamma|!\, G^\bullet(\rho_{\mathcal{G}})}.$$

3.2. Degree sequence

Our goal is to analyse the execution of $\Gamma G^{\bullet}$ so as to obtain information on the degree sequence of random graphs from $\mathcal{G}_n^{\bullet}$. Before we proceed let us make a few important observations. Note that every vertex v different from the root goes through two *phases*. In the first phase, v is generated in a biconnected graph (*i.e.*, in a call to $\Gamma B'$ in the line marked with ($\star\star$)), and has a specific degree. We will also say that v was *born* with this degree. In the second phase, when $\Gamma G^{\bullet}$ is recursively called, a certain number of new biconnected graphs will be attached to v, such that its degree increases by the sum of the degrees of the roots of those graphs. After this, the degree will not change further, so the final degree is the sum of the degrees in the two phases. Hence, to count vertices of a given degree k, we will fix a $1 \leqslant \ell \leqslant k$ and count how many vertices are born with degree ℓ. Let B_ℓ be the number of such vertices. Then, we will compute the fraction of vertices among those B_ℓ that will receive $k-\ell$ neighbours in their second phase. Let us call this fraction $s_{k-\ell}$. The total number of vertices with degree k is then the sum of these numbers over all possible ℓ, namely $\sum_{\ell=1}^{k} B_\ell s_{k-\ell}$.

In order to make these ideas precise we first define suitable generating functions. Let B' denote a random graph from $\mathcal{B}'$, drawn according to the Boltzmann distribution with parameter $x = G^{\bullet}(\rho_{\mathcal{G}})$, and denote by $\mathsf{deg}'(\ell;\mathsf{B}')$ the number of non-root vertices of B' that have degree ℓ. Set

$$I_{\mathcal{B}'}(z) = \sum_{\ell\geqslant 1} \mathbb{E}[\mathsf{deg}'(\ell;\mathsf{B}')]z^{\ell} =: \sum_{\ell\geqslant 1} b_\ell z^{\ell}. \tag{3.2}$$

Now let us turn to $s_{k-\ell}$. Clearly, this value is the probability that a given vertex gets degree exactly $k-\ell$ in the second phase. Let

$$S_{\mathcal{G}}(z) = \sum_{\ell\geqslant 1} s_\ell z^{\ell}$$

be the probability generating function for the degree distribution of a vertex in the second phase. Recall that this degree is the sum of the root degrees of $\mathrm{Po}(\lambda_{\mathcal{G}})$ many graphs $\mathsf{B}'_1, \mathsf{B}'_2, \ldots,$ from $\mathcal{B}'$, drawn independently according to the Boltzmann distribution with parameter $x = G^{\bullet}(\rho_{\mathcal{G}})$. That is, if we define

$$R_{\mathcal{B}'}(z) = \sum_{\ell\geqslant 1} \mathbb{P}\big[\mathsf{rd}(\mathsf{B}') = \ell\big] z^{\ell},$$

where $\mathsf{rd}(\mathsf{B}')$ denotes the root degree of B', and recall that the probability generating function of a $\mathrm{Po}(\lambda)$-distributed random variable is $p(z) = e^{\lambda(z-1)}$, then we see that

$$S_{\mathcal{G}}(z) = e^{\lambda_{\mathcal{G}}(R_{\mathcal{B}'}(z)-1)}. \tag{3.3}$$

Having the functions $I_{\mathcal{B}'}(z)$ and $S_{\mathcal{G}}(z)$ at hand, we now let

$$D_{\mathcal{G}}(z) := I_{\mathcal{B}'}(z)\cdot S_{\mathcal{G}}(z) = I_{\mathcal{B}'}(z)\cdot e^{\lambda_{\mathcal{G}}(R_{\mathcal{B}'}(z)-1)} \quad \text{and} \quad g_k := \sum_{\ell=1}^{k} b_\ell s_{k-\ell} = [z^k]D_{\mathcal{G}}(z). \tag{3.4}$$

Observe that this implies that we have reason to believe that g_k should be equal to the expected fraction of vertices of degree k in a graph G_n that is drawn uniformly at random from $\mathcal{G}_n^\bullet$ (or equivalently, from $\mathcal{G}_n$). The next theorem, which is our main result, describes the conditions under which this intuition is indeed true.

Theorem 3.3. *Let $n \in \mathbb{N}$ and $k = k(n)$ be an integer function of n. Let $\mathcal{G}$ be a nice class of graphs such that $G^\bullet(x)$ is aperiodic, and let $\mathcal{B}$ be the class that contains all biconnected graphs in $\mathcal{G}$. Denote by B' a graph from $\mathcal{B}'$ that is drawn according to the Boltzmann model with parameter $x = G^\bullet(\rho_\mathcal{G})$. Suppose that*

$$\forall 1 \leqslant \ell \leqslant k \;\;:\;\; [z^\ell]I_{\mathcal{B}'}(z) = \mathbb{E}[\mathsf{deg}'(\ell;\, \mathsf{B}')] \;\; \text{is either } 0 \text{ or } \geqslant (n/2)^{-1}\log^4(n/2), \tag{3.5}$$

and set $m = \min\{[z^\ell]I_{\mathcal{B}'}(z) \mid 1 \leqslant \ell \leqslant k \text{ and } [z^\ell]I_{\mathcal{B}'}(z) > 0\}$. Then there is a $C > 0$ such that, for any $(\log n)^{-1/3} < \varepsilon < 1$ and sufficiently large n,

$$\mathbb{P}\big[\mathsf{deg}(k;\, \mathsf{G}_n) \in (1 \pm \varepsilon)\lambda_\mathcal{G} g_k n\big] \geqslant 1 - n^5 e^{-C\varepsilon^2 \frac{g_k}{k} \frac{n}{\max\{1, \log^2(\varepsilon^{-1}m^{-1})\}}},$$

where g_k is given in (3.4).

We split up the proof of the theorem into three parts. First, we show that if the *total* number of vertices of degree ℓ, where $1 \leqslant \ell \leqslant k$, in many independent graphs from $\mathcal{B}'$ is sufficiently concentrated around its expected value, then the conclusion of the above theorem is true.

Lemma 3.4. *There is a $C > 0$ such that the following is true for sufficiently large n. Let $k = k(n)$ be an integer function of n. Let $\mathcal{G}$ be a nice class of graphs such that $G^\bullet(x)$ is aperiodic and let $\mathcal{B}$ be the class that contains all biconnected graphs in $\mathcal{G}$. Suppose that there is a bounded and non-decreasing function $f(\delta) = f(\delta;\, n)$ that has the following property.*

(B) *Let $\mathsf{B}'_1, \ldots, \mathsf{B}'_N$ be graphs drawn independently according to the Boltzmann model for $\mathcal{B}'$ with parameter $x = G^\bullet(\rho_\mathcal{G})$. Set $b_\ell = \mathbb{E}[\mathsf{deg}'(\ell;\, \mathsf{B}'_1)]$. Then, for any $\frac{n}{2} \leqslant N \leqslant \frac{3n}{2}$ and $\delta > (\log N)^{-1/2}$,*

$$\forall\, 1 \leqslant \ell \leqslant k \;\;:\;\; \mathbb{P}\left[\left|\sum_{i=1}^{N} \mathsf{deg}'(\ell;\, \mathsf{B}'_i) - b_\ell N\right| \geqslant \delta b_\ell N\right] \leqslant e^{-\frac{\delta^2}{1+\delta} b_\ell N \cdot f(\delta;\, n)}. \tag{3.6}$$

Then, for every $(\log n)^{-1/3} < \varepsilon < 1$ and sufficiently large n,

$$\mathbb{P}\big[\mathsf{deg}(k;\, \mathsf{G}_n) \in (1 \pm \varepsilon)\lambda_\mathcal{G} g_k n\big] \geqslant 1 - n^5 e^{-C\varepsilon^2 \frac{g_k n}{k} f(\frac{\varepsilon}{10};\, n)}, \tag{3.7}$$

where g_k is given in (3.4).

In order to apply the above lemma we need to check if condition (B) is fulfilled for the class of graphs in question. The following statement provides us with a generic concentration result.

Lemma 3.5. *Let $\mathsf{B}_1, \ldots, \mathsf{B}_N$ be graphs drawn independently from a class $\mathcal{B}$ according to the Boltzmann model with parameter $0 < x < \rho_\mathcal{B}$. There is a $C = C(x) > 0$ such that the following holds. Let $X = X(N) : \mathcal{B} \to \mathbb{N}$ be any function with the property $X(G) \leqslant |G|$*

for every $G \in \mathcal{B}$*. Set* $\mu = \mu(N) = \mathbb{E}[X(\mathsf{B}_i)]$*, and suppose that* $\mu \geqslant \frac{\log^4 N}{N}$*. Then, for any* $\varepsilon = \varepsilon(N) > (\log N)^{-1/2}$ *and sufficiently large* N*,*

$$p_\varepsilon := \mathbb{P}\left[\left|\sum_{i=1}^{N} X(\mathsf{B}_i) - \mu N\right| \geqslant \varepsilon\mu N\right] \leqslant e^{-C\frac{\varepsilon^2}{1+\varepsilon}\frac{\mu N}{(\max\{1,\log(\varepsilon^{-1}\mu^{-1})\})^2}}.$$

An auxiliary tool that we will exploit in the proof of the above lemma is the following statement, which gives a Chernoff-type bound for the total number of vertices in many graphs drawn independently according to the Boltzmann model. Its proof uses standard analytic tools and can be found for completeness in the Appendix.

Lemma 3.6. *Let* $\mathsf{G}_1, \ldots, \mathsf{G}_N$ *be random graphs from a class* $\mathcal{G}$*, drawn independently according to the Boltzmann model with parameter* $0 < x < \rho_{\mathcal{G}}$*. Let* $v = v(x) = \mathbb{E}[|\mathsf{G}_i|]$*. Then, there is a* $C = C(x) > 0$ *such that, for any* $\varepsilon = \varepsilon(N) > 0$*,*

$$\mathbb{P}\left[\sum_{i=1}^{N} |\mathsf{G}_i| \geqslant (1+\varepsilon)vN\right] \leqslant e^{-C(\frac{\varepsilon^2}{1+\varepsilon}vN - 1 - \varepsilon^{-1})}.$$

With all the above tools at hand we are ready to prove our main result for nice classes.

Proof of Theorem 3.3. By applying Lemma 3.1 we obtain that $G^\bullet(\rho_{\mathcal{G}}) < \rho_{\mathcal{B}}$. Let $1 \leqslant \ell \leqslant k$. By applying Lemma 3.5, where we set $x = G^\bullet(\rho_{\mathcal{G}})$ and $X(B') = \mathsf{deg}'(\ell; B')$, we obtain for $\frac{n}{2} \leqslant N \leqslant \frac{3n}{2}$

$$\mathbb{P}\left[\left|\sum_{i=1}^{N} \mathsf{deg}'(\ell, \mathsf{B}_i) - b_\ell N\right| \geqslant \varepsilon b_\ell N\right] \leqslant e^{-C\frac{\varepsilon^2}{1+\varepsilon}b_\ell N \cdot \max\{1, \log(\frac{1}{\varepsilon b_\ell})\}^{-2}},$$

where $b_\ell = [z^\ell] I_{\mathcal{B}'}(z)$. Set

$$f(\varepsilon; n) = C \min_{1 \leqslant \ell \leqslant k} \max\left\{1, \log\left(\frac{1}{\varepsilon b_\ell}\right)\right\}^{-2}.$$

Note that $f(\varepsilon; n) = C\max\{1, \log(\frac{1}{\varepsilon m})\}^{-2}$, and moreover that property (B) in Lemma 3.4 is satisfied with this f. The result follows using the uniform estimate

$$\varepsilon b_\ell \leqslant \varepsilon I_{\mathcal{B}'}(1) = \varepsilon \frac{xB''(x)}{B'(x)} = O(1). \qquad \square$$

What remains is to prove Lemma 3.4 and Lemma 3.5.

Proof of Lemma 3.4. Call a graph $G \in \mathcal{G}_n^\bullet$ *bad* if $\mathsf{deg}(k; G) \notin (1 \pm \varepsilon)\lambda_{\mathcal{G}} g_k n$. As the output distribution of $\Gamma G^\bullet$ is uniform for each n (see Lemma 3.2) and $\mathbb{P}[|\Gamma G^\bullet| = n] = \Theta(n^{-3/2})$, which follows from Lemma 3.1, we infer that

$$\mathbb{P}[\mathsf{G}_n \text{ is bad}] = \frac{\mathbb{P}[(\Gamma G^\bullet \text{ is bad}) \wedge (|\Gamma G^\bullet| = n)]}{\mathbb{P}[|\Gamma G^\bullet| = n]} = O(n^{3/2})\mathbb{P}[(\Gamma G^\bullet \text{ is bad}) \wedge (|\Gamma G^\bullet| = n)]. \tag{3.8}$$

We shall show that the latter probability is at most $n^3 e^{-C\varepsilon^2 \frac{g_k n}{k} f(\frac{\varepsilon}{10})}$, which completes the proof of the lemma.

Recall that every vertex v (different from the root) goes through two phases when constructed by $\Gamma G^\bullet$. In the first phase, v is generated in a biconnected graph (*i.e.*, in a call to $\Gamma B'$), and has a specific degree. We shall also say that v was *born* with this degree. In the second phase, when the sampler is recursively called, a certain number of new biconnected graphs will be attached to v, such that its degree increases by the sum of the degrees of the roots of those graphs. Hence, the degree of a vertex is the sum of two terms: the degree with which it was born, and the number of edges that it acquired in its second phase.

Let $\mathsf{BORN}(\ell;\, n)$ be the set of vertices in the output of $\Gamma G^\bullet$ that were born with degree ℓ, and let $\mathsf{2PHASE}(k-\ell;\, n)$ be the set of vertices that have the property that their degree increased by $k-\ell$ in their second phase. In order to estimate the probability in (3.8), we are going to show that with probability at most $n^2 e^{-C'' \frac{g_k}{k} n f(\frac{\varepsilon}{10})}$,

$$||\mathsf{BORN}(\ell;\, n) \cap \mathsf{2PHASE}(k-\ell;\, n)| - \lambda_{\mathcal{G}} \cdot b_\ell \cdot s_{k-\ell} \cdot n| \geqslant \frac{\varepsilon}{3}\left(\lambda_{\mathcal{G}} b_\ell s_{k-\ell} + \frac{g_k}{k}\right) n. \tag{3.9}$$

Then, by summing over all $1 \leqslant \ell \leqslant k$ (and by adding at most one for the root vertex, which may also have degree k), we obtain for large n that the last probability in (3.8) is at most $kn^2 e^{-C'' \frac{g_k}{k} n f(\frac{\varepsilon}{10})}$, and the proof is complete.

Henceforth let $1 \leqslant \ell \leqslant k$. It will turn out to be very convenient for the analysis to define a slightly modified, but equivalent version (in the sense that the output distributions of both samplers are the same) of the sampler $\Gamma G^\bullet$. Observe that $\Gamma G^\bullet$ makes random choices at two points during its execution: first, when it calculates a random number according to a Poisson distribution in line ($\star$), and second, when it calls the sampler $\Gamma B'$ in line ($\star\star$). We adapt the sampler $\Gamma G^\bullet$ by making the random choices *in advance*, and by providing them as part of the sampler's input. Clearly, this does not alter the probability distribution of the output of the sampler. More precisely, the adapted algorithm $\Gamma G^\bullet(S^{=\ell}, S^{\neq\ell})$ takes as input two *infinite* lists of random values, $S^{=\ell}$ and $S^{\neq\ell}$, which are composed as follows:

$$S^{=\ell} = \big((p_1; b_{1,1}, \ldots, b_{1,p_1}),\ (p_2; b_{2,1}, \ldots, b_{2,p_2}), \ldots, (p_n; b_{n,1}, \ldots, b_{n,p_n}), \ldots\big). \tag{3.10}$$

Here the p_i are independent $\mathrm{Po}(\lambda_{\mathcal{G}})$-distributed variables and all $b_{i,j}$ are independent random graphs according to the Boltzmann distribution for $\mathcal{B}'$ with parameter $x = G^\bullet(\rho_{\mathcal{G}})$ (or equivalently, graphs generated by independent calls to $\Gamma B'(G^\bullet(\rho_{\mathcal{G}}))$). We call every $(p_i; b_{i,1}, \ldots, b_{i,p_i})$ a *block* of the list. $S^{\neq\ell}$ is composed in the same way. $\Gamma G^\bullet(S^{=\ell}, S^{\neq\ell})$ then proceeds as $\Gamma G^\bullet$, and in the two lines ($\star$) and ($\star\star$) uses the values of the two lists, according to the following rules.

(R1) The first block is read from $S^{\neq\ell}$.

(R2) Suppose that the algorithm reaches the line marked with ($*$), and let d_v be the degree of v. In that line, a recursive call to the sampler is initiated. If $d_v = \ell$, then this call will read the next unused block from $S^{=\ell}$, and otherwise from $S^{\neq\ell}$.

Note that for every generated vertex there is precisely one (recursive) call to the sampler. Hence, by construction, the number of vertices in the output of $\Gamma G^\bullet(S^{=\ell}, S^{\neq\ell})$ that were born with degree ℓ equals precisely the number of blocks that the sampler read from $S^{=\ell}$.

Note that the samplers $\Gamma G^\bullet$ and $\Gamma G^\bullet(S^{=\ell}, S^{\neq\ell})$ are equivalent in the sense that

$$\mathbb{P}[\Gamma G^\bullet = \gamma] = \mathbb{P}[\Gamma G^\bullet(S^{=\ell}, S^{\neq\ell}) = \gamma]$$

for every $\gamma \in \mathcal{G}^\bullet$ (where the first probability is taken over the random choices that $\Gamma G^\bullet$ makes, and the second one over the two random lists).

We shall now define two events that allow us to estimate the probability for (3.9).

($\mathfrak{B}$) For $1 \leqslant x \leqslant n$, let $T^{(x)}$ be the list composed of the first x blocks of $S^{=\ell}$, followed by the first $n-x$ blocks of $S^{\neq\ell}$. Let $(p_j^{(x)}; b_{j,1}^{(x)}, \dots)$ be the jth block of it. Moreover, define the random variable

$$B^{(x)} := \sum_{j=1}^{n} \sum_{m=1}^{p_j^{(x)}} \mathsf{deg}'(\ell; b_{j,m}^{(x)}).$$

Then

$$|B^{(x)} - \lambda_{\mathcal{G}} \cdot b_\ell \cdot n| \leqslant \frac{\varepsilon}{9}\left(\lambda_{\mathcal{G}} b_\ell + \frac{g_k}{k}\right)n$$

for every $1 \leqslant x \leqslant n$.

($\mathfrak{R}$) Denote by $(p_j; b_{j,1}, \dots)$ the jth block of $S^{=\ell}$. Then, for every $\ell \leqslant k$, and $n' \in \lambda_{\mathcal{G}} b_\ell n \pm \frac{\varepsilon}{9}(\lambda_{\mathcal{G}} b_\ell + \frac{g_k}{k})n$, the random variable $R_{n'}$ defined below satisfies

$$R_{n'} := \left|\left\{1 \leqslant j \leqslant n' \;\middle|\; \sum_{x=1}^{p_j} \mathsf{rd}(b_{j,x}) = k - \ell\right\}\right| \in s_{k-\ell} n' \pm \frac{\varepsilon}{9}\left(s_{k-\ell} n' + \frac{g_k}{k} n\right).$$

Suppose now that $\mathfrak{B}$ and $\mathfrak{R}$ occur simultaneously. Then the event '(3.9) $\wedge$ $(|\Gamma G^\bullet| = n)$' implies that the sampler $\Gamma G^\bullet(S^{=\ell}, S^{\neq\ell})$ constructed a graph from $\mathcal{G}_n^\bullet$. As for every generated vertex there was precisely one recursive call to $\Gamma G^\bullet(S^{=\ell}, S^{\neq\ell})$, we may deduce that $\Gamma G^\bullet(S^{=\ell}, S^{\neq\ell})$ used in total exactly n blocks out of the lists $S^{=\ell}$ and $S^{\neq\ell}$. Hence there is an $x \geqslant 0$ such that $\Gamma G^\bullet(S^{=\ell}, S^{\neq\ell})$ read precisely x blocks from $S^{=\ell}$, and the remaining ones from $S^{\neq\ell}$. But then $\mathfrak{B}$ implies that the number of vertices born with degree ℓ is in the interval $(1 \pm \frac{\varepsilon}{9}) b_\ell \lambda_{\mathcal{G}} n \pm \frac{\varepsilon}{9}\frac{g_k}{k} n$, as every vertex except the root vertex of the sampled 2-connected rooted graphs is born exactly when this graph is picked by $\Gamma G^\bullet(S^{=\ell}, S^{\neq\ell})$ from one of the two lists. (Note that the root vertex is identified with an already existing vertex). Finally, since $\Gamma G^\bullet(S^{=\ell}, S^{\neq\ell})$ uses, in a recursive call, values from the list $S^{=\ell}$ only if the root of the generated graph is identified with a vertex of degree ℓ, it follows with $\mathfrak{R}$ that the number of vertices born with degree ℓ, and with $k-\ell$ additional adjacent vertices in their second phase, is in

$$\left(1 \pm \frac{\varepsilon}{9}\right) s_{k-\ell} \cdot \left(\lambda_{\mathcal{G}} b_\ell \pm \frac{\varepsilon}{9}\left(\lambda_{\mathcal{G}} b_\ell + \frac{g_k}{k}\right)\right) n \pm \frac{\varepsilon}{9}\frac{g_k}{k} n \subset \lambda_{\mathcal{G}} b_\ell s_{k-\ell} n \pm \frac{\varepsilon}{3}\left(\lambda_{\mathcal{G}} b_\ell s_{k-\ell} + \frac{g_k}{k}\right) n.$$

This is precisely the complement of (3.9). So,

$$\mathbb{P}[(3.9) \wedge (|\Gamma G^\bullet| = n)] = \mathbb{P}[(3.9) \wedge (|\Gamma G^\bullet| = n) \mid \overline{\mathfrak{B}} \text{ or } \overline{\mathfrak{R}}] \cdot \mathbb{P}[\overline{\mathfrak{B}} \text{ or } \overline{\mathfrak{R}}] \leqslant \mathbb{P}[\overline{\mathfrak{B}}] + \mathbb{P}[\overline{\mathfrak{R}}]. \tag{3.11}$$

In the remaining proof we shall bound $\mathbb{P}[\overline{\mathfrak{B}}]$ and $\mathbb{P}[\overline{\mathfrak{R}}]$. To bound the probability for $\overline{\mathfrak{B}}$ we exploit assumption (B) of the lemma. As $\sum_{j=1}^n p_j^{(x)}$ is distributed like $\mathrm{Po}(\lambda_{\mathcal{G}} n)$, Lemma 2.2

implies, for a suitable $C' > 0$,

$$\sum_{j=1}^{n} p_j^{(x)} \in \left(1 \pm \frac{\varepsilon}{90}\right)\lambda_{\mathcal{G}} n \quad \text{with probability at least} \quad 1 - e^{-C'\varepsilon^2 n}.$$

Note that $B^{(x)}$, conditioned on the outcome of $\sum_{j=1}^{n} p_j^{(x)}$, is distributed like the sum of random variables given in (3.6). Let us abbreviate

$$\xi = \frac{\varepsilon}{9}\left(\lambda_{\mathcal{G}} b_\ell + \frac{g_k}{k}\right)n \quad \text{and} \quad \xi' = \xi - \frac{\varepsilon}{90}\lambda_{\mathcal{G}} b_\ell n = \frac{\varepsilon}{10}\left(\lambda_{\mathcal{G}} b_\ell + \frac{10}{9}\frac{g_k}{k}\right)n.$$

We obtain, for large n,

$$\mathbb{P}\big[B^{(x)} \notin \lambda_{\mathcal{G}} b_\ell n \pm \xi\big] \leqslant \sum_{N \in (1 \pm \frac{\varepsilon}{90})\lambda_{\mathcal{G}} n} \mathbb{P}\left[B^{(x)} \notin b_\ell N \pm \xi' \;\Big|\; \sum_{j=1}^{n} p_j^{(x)} = N\right] + e^{-C'\varepsilon^2 n},$$

which is seen to be at most $n \cdot e^{-C''\varepsilon^2 \frac{g_k}{k} n \cdot f(\frac{\varepsilon}{10})}$ as follows, where $C'' > 0$ is suitably chosen. Let us fix any $N \in (1 \pm \frac{\varepsilon}{90})\lambda_{\mathcal{G}} n$, and define α through $\frac{10}{9}\frac{g_k}{k} = \alpha \cdot \lambda_{\mathcal{G}} b_\ell$. Then

$$\xi' = \frac{\varepsilon}{10}(1+\alpha)\lambda_{\mathcal{G}} b_\ell n,$$

and the bounds in (B) imply, for large N with $\delta = \frac{\varepsilon}{10}(1+\alpha) > (\log N)^{-1/2}$,

$$\mathbb{P}\left[B^{(x)} \notin b_\ell N \pm \xi' \;\Big|\; \sum_{j=1}^{n} p_j^{(x)} = N\right] \leqslant \exp\left\{-\frac{1}{10}\frac{\varepsilon^2(1+\alpha)^2}{10+\varepsilon(1+\alpha)} b_\ell N f\left(\frac{\varepsilon}{10}\right)\right\}. \tag{3.12}$$

But the expression

$$\frac{\varepsilon^2(1+\alpha)^2}{10+\varepsilon(1+\alpha)}$$

is easily seen to be of order at least $\varepsilon^2\alpha$ if, say, $\alpha \geqslant 1$, and otherwise, if $0 < \alpha < 1$, it is of order ε^2. This yields with the definition of α the claimed bound. The above discussion implies

$$\mathbb{P}[\overline{\mathfrak{B}}] \leqslant \mathbb{P}\big[\exists 1 \leqslant x \leqslant n : B^{(x)} \notin \lambda_{\mathcal{G}} b_\ell n \pm \xi\big] \leqslant n^2 e^{-C''\varepsilon^2 \frac{g_k}{k} n \cdot f(\frac{\varepsilon}{10})}.$$

Finally, to bound $\overline{\mathfrak{R}}$ we proceed as follows. Recall that

$$\mathbb{P}\left[\sum_{x=1}^{p_j} \mathsf{r}d(b_{j,x}) = k - \ell\right] = [z^{k-\ell}]S_{\mathcal{G}}(z) = s_{k-\ell},$$

where $S_{\mathcal{G}}(z)$ is as defined in (3.3). Hence the distribution of $R_{n'}$ is the same as $\mathrm{Bin}(n', s_{k-\ell})$, and Lemma 2.1 yields with a calculation similar to (3.12) that there is a $C' > 0$ such that

$$\mathbb{P}\left[R_{n'} \notin \left(1 \pm \frac{\varepsilon}{9}\right)s_{k-\ell} n' \pm \frac{\varepsilon}{9}\frac{g_k}{k} n\right] \leqslant e^{-C'\varepsilon^2 \frac{g_k}{k} \cdot n}. \tag{3.13}$$

We readily obtain $\mathbb{P}[\overline{\mathfrak{R}}] \leqslant n \cdot e^{-C'\varepsilon^2 \frac{g_k}{k} \cdot n}$, and the proof is completed with (3.11). □

Proof of Lemma 3.5. Let us first consider the case $\varepsilon\mu \geqslant 2\nu$, where $\nu = \mathbb{E}[|\mathsf{B}_1|] > 0$. As, trivially, $\mu \leqslant \nu$, this can only hold if $\varepsilon \geqslant 2$. In this case, the event '$|\sum_{i=1}^{N} X(\mathsf{B}_i) - \mu N| \geqslant \varepsilon\mu N$'

is thus equivalent to '$\sum_{i=1}^N X(\mathsf{B}_i) \geqslant (1+\varepsilon)\mu N$'. Moreover, as

$$(1+\varepsilon)\mu N \geqslant \frac{\varepsilon\mu N}{2} + \frac{\varepsilon\mu N}{2} \geqslant vN\left(1+\frac{\varepsilon\mu}{2v}\right),$$

we have

$$\mathbb{P}\left[\sum_{i=1}^N X(\mathsf{B}_i) \geqslant (1+\varepsilon)\mu N\right] \leqslant \mathbb{P}\left[\sum_{i=1}^N |\mathsf{B}_i| \geqslant (1+\varepsilon)\mu N\right] \leqslant \mathbb{P}\left[\sum_{i=1}^N |\mathsf{B}_i| \geqslant \left(1+\frac{\varepsilon\mu}{2v}\right)vN\right]$$

and thus obtain the claimed statement by applying Lemma 3.6, where we use for ε the quantity $\frac{\varepsilon\mu}{2v} \geqslant 1$. The remaining proof deals with the case $\varepsilon\mu \leqslant 2v$, from which we deduce with plenty of room to spare that $\varepsilon \leqslant N$.

Before we proceed, note that if $\mathcal{B}$ contains just graphs with the same number of labelled vertices, then the statement follows immediately from the Chernoff bounds. On the other hand, if $\mathcal{B}$ contains graphs with at least two different sizes, then for any $s \in \mathbb{N}$ we have

$$\mathbb{P}[|\mathsf{B}_i| = s] = \Theta\left(\frac{|\mathcal{B}_s|x^s}{s!}\right).$$

This implies for $\rho_{\mathcal{B}} < \infty$ that, up to sub-exponential terms, $|\mathcal{B}_s| \leqslant \rho_{\mathcal{B}}^{-s} s!$, and otherwise $|\mathcal{B}_s| = o(1)^s s!$. We infer that there is a $c > 1$ such that $\mathbb{P}[|\mathsf{B}_i| = s] \leqslant c^{-s}$.

Our first aim is to bound the probability that the total number of vertices that are contained in 'reasonably large' graphs B_i is too large. To make this precise, we first need to define what we mean by 'reasonably large'. Observe that we may assume without loss of generality that $c \leqslant 2$ and hence $\log c < 1$. We then define

$$s_0 := \frac{64}{(\log c)^2} \cdot \max\left\{1, \log\left(\frac{5e}{\varepsilon\mu}\right)\right\}.$$

As we have $\log(ex) \leqslant \sqrt{x}$ for all $x \geqslant 12$, this definition immediately implies that

$$\frac{\log(es_0)}{s_0} \leqslant \frac{1}{\sqrt{s_0}} \leqslant \frac{1}{8}\log(c), \tag{3.14}$$

and similarly

$$\frac{\log\left(\frac{5es_0}{\varepsilon\mu}\right)}{s_0} = \frac{\log(s_0)}{s_0} + \frac{\log\left(\frac{5e}{\varepsilon\mu}\right)}{s_0} \leqslant \begin{cases} \dfrac{\log(s_0)}{s_0} \leqslant \dfrac{1}{8}\log(c), & \text{if } \dfrac{5e}{\varepsilon\mu} \leqslant 1, \\ \dfrac{\log(s_0)}{s_0} + \dfrac{1}{64/(\log c)^2} \leqslant \dfrac{1}{4}\log(c), & \text{otherwise,} \end{cases} \tag{3.15}$$

Having defined s_0 we denote by $\mathcal{E}$ the event that the *total* number of vertices in B_i that contain more than s_0 vertices is greater than $\varepsilon\mu N/5$. In order to bound the probability for $\mathcal{E}$, suppose that there are n B_is with at least s_0 vertices, and suppose that the sum of their sizes is $t \geqslant \varepsilon\mu N/5$. Note that we may assume that $n \leqslant t/s_0$. Observe that there are $\binom{N}{n}$ ways to choose the index set corresponding to the B_is containing at least s_0 vertices, and $\binom{t-ns_0+n-1}{n-1}$ ways to choose the actual size $|\mathsf{B}_i|$ of these B_i (observe that we just have to distribute the 'excess' above s_0). Hence, we can bound

$$\mathbb{P}[\mathcal{E}] \leqslant \sum_{t \geqslant \varepsilon\mu N/5} c^{-t} \cdot \sum_{n \leqslant t/s_0} \binom{N}{n}\binom{t-ns_0+n-1}{n-1}.$$

In order to bound this sum we write $n = \beta t/s_0$ for some $0 < \beta \leqslant 1$ and bound the second binomial coefficient as follows:

$$\binom{t - ns_0 + n - 1}{n-1} \leqslant \binom{t}{n-1} \leqslant \binom{t}{n} \leqslant \left(\frac{et}{n}\right)^n = e^{\beta \log(es_0/\beta)\cdot \frac{t}{s_0}}.$$

As $s_0 \geqslant 3$, one can easily check that $x\ln(s_0/x)$ is increasing for $0 \leqslant x \leqslant 1$. Hence, we have

$$\binom{t - ns_0 + n - 1}{n-1} \leqslant e^{\log(es_0)\cdot \frac{t}{s_0}} \overset{(3.14)}{\leqslant} c^{t/8}.$$

Moreover, for $n \leqslant t/s_0$ and $\varepsilon\mu N/5 \leqslant t \leqslant Ns_0/2$ we trivially have

$$\binom{N}{n} \leqslant \binom{N}{t/s_0} \leqslant \left(\frac{eN}{t/s_0}\right)^{t/s_0} \leqslant \left(\frac{5es_0}{\varepsilon\mu}\right)^{t/s_0} = e^{\log(\frac{5es_0}{\varepsilon\mu})\cdot \frac{t}{s_0}} \overset{(3.15)}{\leqslant} c^{t/4}.$$

On the other hand, if $t \geqslant Ns_0/2$, then also $\binom{N}{n} \leqslant 2^N \leqslant c^{t/4}$. Thus, we deduce that

$$\mathbb{P}[\mathcal{E}] \leqslant \sum_{t \geqslant \varepsilon\mu N/5} c^{-t} \cdot \frac{t}{s_0} \cdot c^{t/8} \cdot c^{t/4}.$$

By the assumptions on the lower bounds on $\varepsilon = \varepsilon(N)$ and $\mu = \mu(N)$, we have that $5t \geqslant \varepsilon\mu N \geqslant (\log N)^2$, from which we deduce that $t \leqslant c^{t/8}$ whenever N is sufficiently large. Hence, we have

$$\mathbb{P}[\mathcal{E}] \leqslant \sum_{t \geqslant \varepsilon\mu N/5} c^{-t/2} = e^{-\Omega(\varepsilon\mu N)}.$$

In other words, with very high probability, the total number of vertices in large B_is is negligible.

With these definitions we are ready to prove the bound for p_ε. Set $Y_i = X(\mathsf{B}_i)\mathbf{1}_{[|\mathsf{B}_i| \leqslant s_0]}$, where $\mathbf{1}_{[\mathfrak{E}]}$ is the indicator function for the event $\mathfrak{E}$. Moreover, set $S' = \sum_{i=1}^N Y_i$ and $S = \sum_{i=1}^N X_i$ and let $M' = \mathbb{M}[S']$ denote the median of S'. Finally, let $E' = \mathbb{E}\big[S'\big]$ and observe that, trivially,

$$\begin{aligned} p_\varepsilon &= \mathbb{P}[|S - \mu N| \geqslant \varepsilon\mu N] \\ &\leqslant \mathbb{P}\left[|S - S'| \geqslant \frac{\varepsilon\mu N}{5}\right] + \mathbb{P}\left[|S' - M'| \geqslant \frac{\varepsilon\mu N}{5}\right] \\ &\quad + \mathbb{P}\left[|M' - E'| \geqslant \frac{\varepsilon\mu N}{5}\right] + \mathbb{P}\left[|E' - \mu N| \geqslant \frac{2\varepsilon\mu N}{5}\right]. \end{aligned}$$

We will now bound each of the four terms. Firstly, recall that by assumption $X(G) \leqslant |G|$ for all $G \in \mathcal{B}$. Hence, we have

$$\mathbb{P}\left[|S - S'| \geqslant \frac{\varepsilon\mu N}{5}\right] \leqslant \mathbb{P}[\mathcal{E}] = e^{-\Omega(\varepsilon\mu N)}$$

and thus also

$$\mu N - E' = \mathbb{E}[S - S'] \leqslant \mathbb{E}[S - S' \mid \overline{\mathcal{E}}] + N \cdot \mathbb{P}[\mathcal{E}] \leqslant \frac{2\varepsilon\mu N}{5},$$

with room to spare. We infer that

$$\mathbb{P}\left[|E' - \mu N| \geqslant \frac{2\varepsilon\mu N}{5}\right] = 0.$$

In order to bound the remaining two terms we apply Theorem 2.3 with respect to the variables $Y_1, \ldots, Y_N$ and $S' = \sum_{i=1}^{N} Y_i$. By the definition of the Y_i and the assumption on $X(G)$, we immediately deduce that the effect of the ith coordinate on S' is bounded from above by s_0. In other words, we can set $c_i = s_0$ in Theorem 2.3. Moreover, if $S' \geqslant t$, then there is an index set $\mathcal{T}$ containing at most t entries such that $\sum_{i\in\mathcal{T}} Y_i \geqslant t$. Hence, we can apply Theorem 2.3 with $\psi(x) = x \cdot s_0^2$ to deduce that

$$\mathbb{P}\big[|S' - M'| \geqslant t\big] \leqslant 4\exp\left\{-\frac{t^2}{4(M'+t)s_0^2}\right\}. \tag{3.16}$$

Below we argue that $|E' - M'| = O(s_0\sqrt{E'})$, which has the following consequences. Observe that the definition of s_0 implies that there exists a constant $C' = C'(x)$ such that $s_0 \leqslant C' \cdot \max\{1, \log(\varepsilon^{-1}\mu^{-1})\}$. As by our assumptions we have $\varepsilon\mu \geqslant 1/N$, we deduce $s_0 \leqslant C' \log N$ and thus $s_0\sqrt{E'} = o(\varepsilon\mu N)$. This implies that

$$\mathbb{P}\left[|M' - E'| \geqslant \frac{1}{5}\varepsilon\mu N\right] = 0, \quad \text{whenever } N \text{ is sufficiently large,}$$

and, with room to spare, also $M' \leqslant 2\mu N$. Hence, we deduce from (3.16) that

$$\mathbb{P}\left[|S' - M'| \geqslant \frac{1}{5}\varepsilon\mu N\right] \leqslant 4\exp\left\{-\frac{1}{200}\frac{\varepsilon^2}{(1+\varepsilon)s_0^2}\mu N\right\}.$$

The proof of $|E' - M'| = O(s_0\sqrt{E'})$ is actually very similar to the one in [15, p. 42], where it is performed for the special case $\psi(x) = x$; we thus sketch only the most important ideas. First, note that

$$|E' - M'| \leqslant \mathbb{E}[|S' - M'|] \leqslant \sum_{t\geqslant 1} \mathbb{P}[|S' - M'| \geqslant t].$$

By using (3.16), this is easily seen to be of order at most $s_0\sqrt{M'}$. Finally, as $M'/2 \leqslant M'\mathbb{P}[S' \geqslant M'] \leqslant E'$, we obtain $|E' - M'| = O(s_0\sqrt{E'})$, as desired. □

4. Applications

The main ingredient for a successful application of Theorem 3.3 to a specified nice class of graphs is the computation of the following two functions: first, $R_{\mathcal{B}'}(z)$, which is the probability generating function for the root degree of a graph from $\mathcal{B}'$, drawn according to the Boltzmann model, and second, $I_{\mathcal{B}'}(z)$, whose ℓth coefficient is the expected number of non-root vertices of degree ℓ in such a random graph. Before we proceed to specific applications, we demonstrate that for many natural classes of graphs it is sufficient just to determine $R_{\mathcal{B}'}(z)$, and $I_{\mathcal{B}'}(z)$ is then readily obtained.

Before we describe the classes we will consider, let us fix some notation. Let $\mathcal{B}'$ be a class consisting of labelled derived graphs. Let B' be a graph from $\mathcal{B}'$, drawn according to the Boltzmann distribution with parameter x, and denote by $\mathsf{rd}(\mathsf{B}')$ the degree of the root vertex of B'. Then we write $R_{\mathcal{B}'}(z;x)$ for the probability generating function of $\mathsf{rd}(\mathsf{B}')$, and $I_{\mathcal{B}'}(z;x)$ for the function whose ℓth coefficient is equal to $\mathbb{E}[\mathsf{deg}'(\ell;\mathsf{B}')]$, *i.e.*,

$$R_{\mathcal{B}'}(z;x) := \sum_{k\geqslant 0}\mathbb{P}[\mathsf{rd}(\mathsf{B}')=k]z^k \quad\text{and}\quad I_{\mathcal{B}'}(z;x) := \sum_{k\geqslant 0}\mathbb{E}[\mathsf{deg}'(\ell;\mathsf{B}')]z^k.$$

We will use this notation throughout the paper without further reference.

Definition 2. Let $\mathcal{B}'$ be any class of labelled derived graphs. $\mathcal{B}'$ is called *isothermic* if, for any n, any $0\leqslant k<n$, and any vertex $v\in\{1,\dots,n\}$, we have

$$\mathbb{P}[\mathsf{deg}_{\mathsf{B}'_n}(v)=k]=\mathbb{P}[\mathsf{rd}(\mathsf{B}'_n)=k],$$

where B'_n denotes a graph drawn uniformly at random from $\mathcal{B}'_n$.

In words, isothermic random graphs are *symmetric* with respect to the degree distribution of their vertices. It is easily verified that for example labelled 2-connected outerplanar and series-parallel graphs form isothermic classes, as every relabelling of the vertex set yields again a member of the corresponding graph class.

The next lemma provides us with an explicit relation for $R_{\mathcal{B}'}(z;x)$ and $I_{\mathcal{B}'}(z;x)$. Here we write $|B'|$ for the number of non-root vertices of any $B'\in\mathcal{B}'$.

Lemma 4.1. *Let $\mathcal{B}'$ be an isothermic class of derived graphs and let B' be a random graph drawn according to the Boltzmann distribution with parameter $0\leqslant x<\rho_{\mathcal{B}'}$. Then*

$$I_{\mathcal{B}'}(z;x)=x\frac{\partial}{\partial x}R_{\mathcal{B}'}(z;x)+\mathbb{E}[|\mathsf{B}'|]R_{\mathcal{B}'}(z;x).$$

Proof. From the definition of $I_{\mathcal{B}'}(z;x)$ and the assumption that $\mathcal{B}'$ is isothermic we infer that

$$I_{\mathcal{B}'}(z;x)=\sum_{\ell\geqslant 0}z^\ell\cdot\sum_{n\geqslant 0}\frac{b'_n x^n}{n!B'(x)}\cdot n\mathbb{P}[\mathsf{rd}(\mathsf{B}'_n)=\ell].$$

Note that

$$\frac{nx^{n-1}}{B'(x)}=\frac{\partial}{\partial x}\left(\frac{x^n}{B'(x)}\right)+\frac{x^n\frac{\partial}{\partial x}B'(x)}{B'(x)^2}.$$

We obtain

$$\begin{aligned}I_{\mathcal{B}'}(z;x)&=x\sum_{\ell\geqslant 0}z^\ell\sum_{n\geqslant 0}\frac{b'_n}{n!}\cdot\left(\frac{\partial}{\partial x}\left(\frac{x^n}{B'(x)}\right)+\frac{x^n\frac{\partial}{\partial x}B'(x)}{B'(x)^2}\right)\mathbb{P}[\mathsf{rd}(\mathsf{B}'_n)=\ell]\\&=x\sum_{\ell\geqslant 0}z^\ell\cdot\left([z^\ell]\frac{\partial}{\partial x}R_{\mathcal{B}'}(z;x)+\frac{\frac{\partial}{\partial x}B'(x)}{B'(x)}[z^\ell]R_{\mathcal{B}'}(z;x)\right).\end{aligned}$$

The proof concludes with the observation

$$\mathbb{E}[|\mathsf{B}'|]=\frac{x\frac{\partial}{\partial x}B'(x)}{B'(x)}.\qquad\square$$

This section closes with two immediate applications of Theorem 3.3 and of the concept of isothermic classes: Cayley trees and cactus graphs.

4.1. Labelled trees

Let $\mathcal{T}$ denote the class of all labelled trees, and denote by $\mathcal{B}$ the set of graphs in $\mathcal{T}$ that are biconnected. It is well known (see, *e.g.*, [10]) that the egf enumerating $\mathcal{T}^\bullet$ has a unique dominant singularity at $\rho_{\mathcal{T}} = e^{-1}$ and that $T^\bullet(\rho_{\mathcal{T}}) = 1$.

Clearly, $\mathcal{B}$ consists of only one graph, *i.e.*, a single edge. Thus $R_{\mathcal{B}'}(z;x) = z$, and by Lemma 4.1 also $I_{\mathcal{B}'}(z;x) = z$. Hence, we can immediately apply Theorem 3.3 and obtain, for a tree T_n drawn uniformly at random from $\mathcal{T}_n$,

$$\mathbb{P}\big[\mathsf{deg}(k;\mathsf{T}_n) \in (1\pm\varepsilon)t_k n\big] \geqslant 1 - n^5 e^{-C\frac{\varepsilon^2}{\log^2(\varepsilon^{-1})}\frac{t_k}{k}n}, \quad \text{where } \; t_k = [z^k]ze^{z-1} = \frac{1}{e(k-1)!}.$$

In other words, we obtain exponentially small tail bounds for the number of vertices of degree $k \leqslant (1-o(1))\frac{\log n}{\log\log n}$ in T_n. Note that the maximum degree of T_n is $\sim \frac{\log n}{\log\log n}$, by Moon's result [17]. Consequently, Theorem 3.3 provides us with a concentration result for all desirable values of k.

4.2. Cactus graphs

We say that a labelled connected graph is a *cactus* if all its maximal biconnected components are cycles or edges. Let $\mathcal{C}$ be the class that contains all cactus graphs, and let $\mathcal{B}$ be the class that contains all labelled cycles and a single edge. In this subsection we will use Theorem 3.3 to show large deviation estimates for the number of vertices of degree k in graph C_n that is drawn uniformly at random from $\mathcal{C}_n$.

In [20] an explicit expression for $C_0^\bullet := \mathcal{C}^\bullet(\rho_{\mathcal{C}}) \doteq 0.4563$ was derived, and it was shown that $\rho_{\mathcal{B}}B''(\rho_{\mathcal{B}}) > 1$. Moreover, it is easily seen that

$$B(x) = -\frac{1}{2}\log(1-x) + \frac{1}{4}x^2 - \frac{1}{2}x \quad \text{and} \quad B'(x) = \frac{x(2-x)}{2(1-x)}.$$

With those facts we readily obtain that

$$R_{\mathcal{B}'}(z;x) = \frac{1}{B'(x)}\big(xz + (B'(x)-x)z^2\big) = \frac{z(2-2x+xz)}{2-x}.$$

Additionally, Lemma 4.1 implies that

$$I_{\mathcal{B}'}(z;x) = \frac{2(1-x)}{2-x}z + \frac{x}{1-x}z^2.$$

With this information at hand, we can immediately apply Theorem 3.3 and obtain

$$\mathbb{P}\big[\mathsf{deg}(k;\mathsf{C}_n) \in (1\pm\varepsilon)\lambda_{\mathcal{C}}c_k n\big] \geqslant 1 - n^5 e^{-C\frac{\varepsilon^2}{(\log(\varepsilon^{-1})+2)^2}\frac{c_k}{k}n}$$
$$\text{and} \quad c_k = [z^k]I_{\mathcal{B}'}(z;C_0^\bullet)e^{\lambda_{\mathcal{C}}(R_{\mathcal{B}'}(z;C_0^\bullet)-1)}.$$

The asymptotic form of the c_k's can be derived with some technical work. We omit the details and present just the final result. There are constants $C_1, C_2, C_3 > 0$ such that

$$c_k = (C_1 + o(1)) \cdot k^{-k/2} \cdot C_2^k \cdot C_3^{\sqrt{k}} \cdot k^{1/2}.$$

In other words, we obtain exponentially small tail bounds for the number of vertices of degree $k \leqslant (2 - o(1))\frac{\log n}{\log\log n}$ in C_n. Note that the maximum degree of C_n is $\sim 2\frac{\log n}{\log\log n}$, by the results in [20]. Consequently, Theorem 3.3 provides us also in this case with a concentration result for all desirable values of k.

5. Outerplanar graphs

In this section we determine the degree sequence of large random outerplanar graphs. Let $\mathcal{O}$ be the class of all labelled connected outerplanar graphs, and $\mathcal{B}$ the class of labelled biconnected outerplanar graphs. Bodirsky, Giménez, Kang and Noy [4] showed that $\mathcal{O}$ is nice. We will argue that the remaining preconditions of Theorem 3.3 are fulfilled. In particular, we show the following theorem, where '$\doteq$' means that the quantity on the right side is truncated to the digits shown.

Theorem 5.1. *There are explicitly given constants $\mu \doteq 0.38081$ and $\lambda_{\mathcal{O}} \doteq 0.22327$ such that the class of labelled connected outerplanar graphs satisfies the preconditions of Theorem 3.3 for any $k \leqslant \log_{1/\mu} n - 5\log_{1/\mu}\log n$. Consequently, there is a $C > 0$ such that if we denote by O_n a random graph from $\mathcal{O}_n$, then we have for any $0 < \varepsilon < 1$ that*

$$\mathbb{P}\big[\mathsf{deg}(k;\mathsf{O}_n) \in (1 \pm \varepsilon)o_k n\big] \geqslant 1 - e^{-C\frac{\varepsilon^2}{(\log(\varepsilon^{-1})+k)^2}\frac{o_k}{k}n},$$

and $o_k = [z^k]\lambda_{\mathcal{O}} D_{\mathcal{O}}(z)$, where $D_{\mathcal{O}}(z)$ is given by (3.4), $R_{\mathcal{B}'}(z)$ is given by Lemma 5.4 below, and then $I_{\mathcal{B}'}(z)$ by Lemma 4.1. Moreover,

$$o_1 \doteq 0.13659, \quad o_2 \doteq 0.28753, \quad o_3 \doteq 0.24287,$$
$$o_4 \doteq 0.15507, \quad o_5 \doteq 0.08743, \ldots$$

and, for large k,

$$o_k = (c_1 + o(1)) \cdot \mu^k \cdot e^{c_2\sqrt{k}} \cdot k^{1/4},$$

where $c_1, c_2 > 0$ are analytically given.

The formula in the above theorem strongly indicates that the maximum degree $\Delta(\mathsf{O}_n)$ of a random outerplanar graph O_n is roughly $\log_{1/\mu} n$. Unfortunately, our current techniques are not strong enough to prove this, although we come very close to this value. We thus formulate it as a conjecture.

Conjecture 5.2. *For any $\varepsilon > 0$, we have $\lim_{n\to\infty} \mathbb{P}[\Delta(\mathsf{O}_n) \in (1 \pm \varepsilon)\log_{1/\mu} n] = 1$.*

In order to prove the theorem we are first going to derive explicit expressions for the functions $D_{\mathcal{O}}(z)$, $R_{\mathcal{B}'}(z)$ and $I_{\mathcal{B}'}(z)$, which encode the degree distribution in large random outerplanar graphs, and the root degree and degree distributions in random graphs from the Boltzmann model, respectively. Then, we will derive appropriate asymptotics for the coefficients of those functions, which will enable us to apply Theorem 3.3.

First we shall determine the function $R_{\mathcal{B}'}(z)$. In order to achieve this we will investigate an auxiliary graph class. Note that an outerplanar graph is 2-connected if and only if

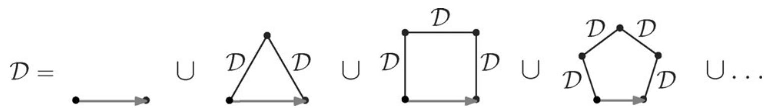

Figure 1. Recursive decomposition of the class of dissections.

it has a unique Hamilton cycle. Hence any 2-connected outerplanar graph is in fact equivalent to a *dissection of a convex polygon*, where the boundary of the polygon is the (unique) Hamilton cycle. Therefore, the class of 2-connected outerplanar graphs is essentially equivalent to the class of dissections of convex polygons. Below we make this connection explicit.

Let P be a convex polygon with n unlabelled vertices and fix an edge e of P. A dissection of P is then either this single edge or an ordered sequence of $\ell \geqslant 2$ dissections along the face containing e, where $\ell - 1$ pairs of vertices are identified: see Figure 1. Thus the ordinary generating function $D(x)$ for polygon dissections, which are rooted at an edge e of the outer face, where x marks the vertices, satisfies

$$D(x) = x^2 + \frac{D(x)^2}{x} + \frac{D(x)^3}{x^2} + \cdots = \frac{x}{4}\left(1 + x - \sqrt{x^2 - 6x + 1}\right). \tag{5.1}$$

Henceforth we are going to make use of the following simple proposition. A similar statement was proved in [4], but for the sake of completion we give here a self-contained proof.

Proposition 5.3. *The* egf *enumerating rooted labelled* 2*-connected outerplanar graphs satisfies*

$$B^{\bullet}(x) = xB'(x) = \frac{1}{2}\left(D(x) + x^2\right),$$

where $D(x)$ is the ordinary generating function *enumerating unlabelled edge-rooted dissections.*

Proof. Let

$$B^{\bullet}(x) = \sum_{n \geqslant 2} \frac{b_n^{\bullet}}{n!} x^n \quad \text{and} \quad D(x) = \sum_{n \geqslant 2} d_n x^n.$$

The claim can be seen as follows. For $n \geqslant 3$, every edge-rooted dissection gives rise to $(n-1)!/2$ distinct 2-connected outerplanar graphs, and therefore to $n!/2$ distinct rooted biconnected outerplanar graphs. For the special case $n = 2$, $d_2 = 1$ and $b_2^{\bullet} = 2$. With this we obtain that

$$B^{\bullet}(x) = x^2 + \sum_{n \geqslant 3} \frac{d_n}{2} x^n = x^2 + \frac{1}{2} \sum_{n \geqslant 2} d_n x^n - \frac{1}{2} d_2 x^2 = \frac{1}{2}(D(x) + x^2). \qquad \square$$

We are now ready to derive an explicit expression for $R_{\mathcal{B}'}(z)$.

Lemma 5.4. *Let* B' *be a graph from* $\mathcal{B}'$*, drawn according to the Boltzmann model with parameter* $0 \leqslant x \leqslant \rho_{\mathcal{B}}$*. Then*

$$\mathbb{P}[\mathsf{rd}(\mathsf{B}') = \ell] = [z^\ell]R_{\mathcal{B}'}(z\,;\,x) = [z^\ell]\frac{x}{B'(x)} \cdot \frac{(2B'(x)-x)z + (x - B'(x))z^2}{(2B'(x)-x) - 2(B'(x)-x)z}.$$

Proof. Let D be a dissection, drawn according to the Boltzmann model with parameter x. Following the recursive decomposition for $\mathcal{D}$, D can be constructed according to the compilation rules for Boltzmann samplers in [9] with the following simple algorithm:

$\Gamma D(x)$: choose a random value $L \in \mathbb{N}$ such that $\mathbb{P}[L=1] = \frac{x^2}{D(x)}$
and $\mathbb{P}[L=s] = \left(\frac{D(x)}{x}\right)^{s-1}$ for $s \geqslant 2$
if $\ell = 1$ **return** a single edge
else $\gamma_1 \leftarrow \Gamma D(x), \ldots, \gamma_{L-1} \leftarrow \Gamma D(x)$ (independent recursive calls)
return a dissection composed out of $\gamma_1, \ldots, \gamma_{L-1}$, as in Figure 1

We refer the reader also to [3], where this algorithm is discussed in great detail. Let the root vertex of D be the tail of the root edge, as indicated in Figure 1. We immediately obtain

$$\mathbb{P}[\mathsf{rd}(\mathsf{D}) = 1] = \frac{x^2}{D(x)},$$

as $\mathsf{rd}(\mathsf{D}) = 1$ if and only if $\Gamma D(x)$ chooses $L = 1$. Moreover, D has root degree $k \geqslant 2$ if and only if $L \geqslant 2$ and $\mathsf{rd}(\gamma_1) = k - 1$. A simple inductive argument then shows that

$$\mathbb{P}[\mathsf{rd}(\mathsf{D}) = k] = \frac{x^2}{D(x)}\left(1 - \frac{x^2}{D(x)}\right)^{k-1}.$$

In order to study the root degree distribution of random graphs from $\mathcal{B}'$, we design in the next step a Boltzmann sampler $\Gamma B'(x)$. We start by describing a sampler $\Gamma B^\bullet(x)$ that generates *rooted* 2-connected outerplanar graphs. Let $\mathrm{Ber}(p)$ be a Bernoulli variable that obtains the value 1 with probability p. Then

$\Gamma B^\bullet(x)$: **if** $(\mathrm{B}er(\frac{x^2}{2B^\bullet(x)}) = 1)$ **then** $\gamma \leftarrow$ a single rooted edge
else $\gamma \leftarrow \Gamma D(x)$
root γ at the tail of its root edge
label the vertices of γ uniformly at random
return γ

Now we prove that

$$\mathbb{P}[\Gamma B^\bullet(x) = \gamma] = \frac{x^{|\gamma|}}{|\gamma|!B^\bullet(x)}$$

for $\gamma \in \mathcal{B}^\bullet$, *i.e.*, $\Gamma B^\bullet(x)$ is a Boltzmann sampler for $\mathcal{B}^\bullet$. We distinguish two cases. If γ is a single rooted edge, then it can be generated in two ways: either the Bernoulli variable evaluates to 1, or otherwise $\Gamma D(x)$ outputs an edge. By exploiting Proposition 5.3 and by abbreviating $t(x) = \frac{x^2}{2B^\bullet(x)}$, we obtain

$$\mathbb{P}[\Gamma B^\bullet(x) = \gamma] = \frac{1}{2!} \cdot \left(t(x) + (1 - t(x)) \frac{x^2}{D(x)} \right) = \frac{1}{2!} \frac{x^2}{B^\bullet(x)}. \tag{5.2}$$

On the other hand, if $|\gamma| \geqslant 3$, then γ can only be generated if the Bernoulli variable evaluates to 0. Furthermore, there are either two edge-rooted dissections, which if correctly labelled yield γ, *or* there is a unique edge-rooted dissection which can be labelled in two different ways to obtain γ. Hence,

$$\mathbb{P}[\Gamma B^\bullet(x) = \gamma] = 2 \cdot \frac{1}{|\gamma|!} \cdot (1 - t(x)) \cdot \frac{x^{|\gamma|}}{D(x)} = \frac{x^{|\gamma|}}{|\gamma|! B^\bullet(x)}.$$

Hence, $\Gamma B^\bullet(x)$ is indeed a Boltzmann sampler for $\mathcal{B}^\bullet$.

Note that we can exploit $\Gamma B^\bullet(x)$ to obtain random graphs from $\mathcal{B}'$ by removing the label from the root vertex of its output γ, and by relabelling the other vertices such that they have labels from $\{1, \ldots, |\gamma| - 1\}$. This can be obviously done in a unique way, *e.g.*, by relabelling them while preserving their order. Now, as for every $\gamma' \in \mathcal{B}'$ there are precisely $|\gamma'| + 1$ distinct graphs $\gamma_1, \gamma_2, \ldots$ in $\mathcal{B}^\bullet$ such that we can obtain by the above procedure γ', we infer that

$$\mathbb{P}[\gamma' \text{ was drawn}] = \sum_{i=1}^{|\gamma'|+1} \mathbb{P}[\Gamma B^\bullet(x) = \gamma_i] = \sum_{i=1}^{|\gamma'|+1} \frac{x^{|\gamma'|+1}}{(|\gamma'| + 1)! B^\bullet(x)} = \frac{x^{|\gamma'|}}{|\gamma'|! B'(x)}.$$

It follows that the root degree distributions of $\Gamma B^\bullet(x)$ and $\Gamma B'(x)$ are the same. But then, $\mathbb{P}[\mathsf{rd}(\Gamma B'(x)) = 1] = \mathbb{P}[\Gamma B^\bullet(x) \text{ outputs a rooted edge}] = 2t(x)$, by the same argument as in (5.2). Moreover, $\mathsf{rd}(\Gamma B'(x)) = k$ if and only if the Bernoulli variable evaluates to 0 and $\Gamma D(x)$ returns a dissection with root degree k. Hence,

$$\mathbb{P}[\mathsf{rd}(\Gamma B'(x)) = k] = (1 - t(x)) \frac{x^2}{D(x)} \left(1 - \frac{x^2}{D(x)} \right)^{k-1}.$$

The proof is completed by summing up these expressions and using Proposition 5.3. $\square$

Proof of Theorem 5.1. As already discussed previously, in [4] it was shown that the class $\mathcal{O}$ is nice. What remains is to check (3.5). Let us abbreviate $O_0^\bullet = O^\bullet(\rho_{\mathcal{O}})$. Note that the class $\mathcal{B}'$ is isothermic. By applying Lemma 4.1 we obtain

$$I_{\mathcal{B}'}(z) = O_0^\bullet R'_{\mathcal{B}'}(z) + \frac{O_0^\bullet B''(O_0^\bullet)}{B'(O_0^\bullet)} \cdot R_{\mathcal{B}'}(z),$$

where

$$R_{\mathcal{B}'}(z) = R_{\mathcal{B}'}(z;\, O_0^\bullet), \quad R'_{\mathcal{B}'}(z) = \frac{\partial}{\partial x} R_{\mathcal{B}'}(z;\, x)|_{x = O_0^\bullet},$$

and all other derivatives are taken with respect to x. A straightforward analysis shows that $[z^\ell] R_{\mathcal{B}'}(z) = \Theta(\mu^\ell)$ and $[z^\ell] R'_{\mathcal{B}'}(z) = \Theta(\ell \mu^\ell)$, where

$$\mu = \frac{2(B'(O_0^\bullet) - O_0^\bullet)}{2B'(O_0^\bullet) - O_0^\bullet}.$$

Hence,

$$[z^\ell]I_{\mathcal{B}'}(z) = \Theta(\ell\mu^\ell),$$

which implies that Theorem 3.3 can be applied whenever, say, $k \leqslant \log_{1/\mu} n - 5\log_{1/\mu}\log n$. This gives us that $D_{\mathcal{O}}(z) = I_{\mathcal{B}'}(z) \cdot e^{\lambda_{\mathcal{O}}(R_{\mathcal{B}'}(z)-1)}$, where $\lambda_{\mathcal{O}} = B'(O_0^\bullet)$. Furthermore we observe that in general for a function $f(z)$ we have $[z^k]f(a\cdot z) = a^k \cdot [z^k]f(z)$, and hence we can apply Lemma 2.4 to calculate the asymptotic form of o_k for large values of k.

The numerical values claimed in the theorem were calculated with the help of Maple, by using the explicit expression for the function $B'(x)$ from Proposition 5.3 and the value of $O_0^\bullet$ from [4]. □

6. Series-parallel graphs

In this section we determine the degree sequence of graphs drawn uniformly at random from the class $\mathcal{SP}$ of labelled connected series-parallel (SP) graphs. For the remainder of the section we will denote by $\mathcal{B}$ the class of labelled biconnected SP graphs. Bodirsky, Giménez, Kang and Noy [4] showed that $\mathcal{SP}$ is nice. As in the previous section, we now argue that the remaining preconditions of Theorem 3.3 are fulfilled, and prove the following statement.

Theorem 6.1. *There are explicitly given constants* $\mu \doteq 0.75041$ *and* $\lambda_{\mathcal{SP}} \doteq 0.14937$ *such that the class of labelled connected series-parallel graphs satisfies the preconditions of Theorem 3.3 for any* $k \leqslant \log_{1/\mu} n - 20\log_{1/\mu}\log n$. *Consequently, there is a* $C > 0$ *such that if we denote by* SP_n *a random graph from* $\mathcal{SP}_n$*, then we have for any* $0 < \varepsilon < 1$ *that*

$$\mathbb{P}\big[\mathsf{deg}(k;\, \mathsf{SP}_n) \in (1\pm\varepsilon)sp_k n\big] \geqslant 1 - e^{-C\frac{\varepsilon^2}{(\log(\varepsilon^{-1})+k)^2}\frac{sp_k}{k}n},$$

and $sp_k = [z^k]D_{\mathcal{SP}}(z)$*, where* $D_{\mathcal{SP}}(z)$ *is given by* (3.4)*,* $R_{\mathcal{B}'}(z)$ *is given by Lemma 6.3 below, and then* $I_{\mathcal{B}'}(z)$ *by Lemma 4.1. Moreover, we have*

$$sp_1 \doteq 0.11021, \quad sp_2 \doteq 0.35637, \quad sp_3 \doteq 0.22335,$$
$$sp_4 \doteq 0.12576, \quad sp_5 \doteq 0.07172, \ldots$$

and, for large k*,*

$$sp_k = (c_1 + o(1)) \cdot \mu^k \cdot k^{-3/2},$$

where c_1 *is analytically given.*

As in the case of outerplanar graphs, the above formula for sp_k suggests the following conjecture for the maximum degree $\Delta(\mathsf{SP}_n)$ of SP_n.

Conjecture 6.2. *For any* $\varepsilon > 0$*, we have* $\lim_{n\to\infty}\mathbb{P}[\Delta(\mathsf{SP}_n) \in (1\pm\varepsilon)\log_{1/\mu} n] = 1$.

In order to prove the theorem we are first going to derive explicit expressions for the functions $D_{\mathcal{SP}}(z)$, $R_{\mathcal{B}'}(z)$ and $I_{\mathcal{B}'}(z)$, and then we will derive appropriate asymptotics for the coefficients of those functions. We begin with the function $R_{\mathcal{B}'}(z)$. Before we proceed, let us introduce an auxiliary graph class, which plays an important role in the decomposition of 2-connected series-parallel graphs. Following [19, 4], we define a *network* as a connected graph with two distinguished vertices, called the left and the right *pole*, such that adding the edge between the poles the resulting (multi)graph is 2-connected. Let $\mathcal{D}$ be the class of series-parallel networks, such that $\mathcal{D}_n$ contains all networks with n non-pole vertices. We write for brevity $\mathcal{D}_0 \equiv e$ for the network consisting of a single edge. Let $\vec{\mathcal{B}}$ be the class containing all graphs in $\mathcal{B}$ rooted at any of their edges, where the root edge is oriented. Then, due to the definition of $\mathcal{D}$, the classes $\mathcal{B}$ and $\mathcal{D}$ are related as follows (see also [19]):

$$(\mathcal{D}+1) \times \mathcal{Z}^2 \times e = (1+e) \times \vec{\mathcal{B}}. \tag{6.1}$$

Although this decomposition can be used to obtain detailed information about the generating function enumerating $\mathcal{B}$ (see, *e.g.*, [1]), as well as the degree sequence of a 'typical' graph from $\vec{\mathcal{B}}$, it turns out that it is quite involved to derive from it information about the degree sequence of a random graph from $\mathcal{B}$. This difficulty is mainly due to the fact that the number of ways to root a graph at an edge varies for graphs of the same size (with respect to the number of vertices), and would require to perform a very laborious integration. We attack this problem differently: we exploit a very general result by Chapuy, Fusy, Kang and Shoilekova [6], which allows us to decompose families of 2-connected graphs in a direct combinatorial way (again based on networks), but avoiding the often complicated and intractable integration steps.

Given this, the distribution of the root degree of random graphs from $\mathcal{B}'$ is as follows.

Lemma 6.3. *Let* B' *be a graph drawn from* $\mathcal{B}'$ *according to the Boltzmann distribution with parameter* $0 \leqslant x \leqslant \rho_{\mathcal{B}'}$. *Then*

$$\mathbb{P}[\mathsf{rd}(\mathsf{B}') = \ell] \;=\; [z^\ell]R_{\mathcal{B}'}(z;x) \;:=\; [z^\ell]\frac{R_{\mathcal{D}}(z;x)(xD(x)^2R_{\mathcal{D}}(z;x)-2)}{xD(x)^2-2}, \tag{6.2}$$

where $D(x)$ *is the* egf *enumerating series-parallel networks, and* $R_{\mathcal{D}}(z)$ *satisfies*

$$R_{\mathcal{D}}(z;x) = \frac{1}{D(x)}\left(-1+(1+z)\left(\frac{1+D(x)}{2}\right)^{R_{\mathcal{D}}(z;x)}\right). \tag{6.3}$$

We defer the proof of this and the following corollary to Section 6.1 and use the remainder of this section to explain the main ideas of our approach. A further ingredient that is needed for the application of Theorem 3.3 is the evaluation of the coefficients of the function $I_{\mathcal{B}'}(z;x)$, whose ℓth coefficient is the expected number of non-root vertices of degree ℓ in a random graph from $\mathcal{B}'$, drawn according to the Boltzmann model with parameter x. As the class $\mathcal{B}'$ is easily seen to be isothermic, we can apply Lemma 4.1 to obtain an explicit expression for $I_{\mathcal{B}'}(z;x)$ in terms of $\frac{\partial}{\partial x}R_{\mathcal{B}'}(z;x)$. The following corollary gives the singular expansions of $R_{\mathcal{B}'}(z;x)$ and $\frac{\partial}{\partial x}R_{\mathcal{B}'}(z;x)$ that will become very handy in the proof of Theorem 6.1, where we will extract information about the growth rate of the coefficients.

Corollary 6.4. *Let $0 \leqslant x \leqslant \rho_{\mathcal{B}'}$. Then $R_{\mathcal{B}'}(z;x)$ and $I_{\mathcal{B}'}(z;x)$ have a unique dominant singularity $\rho(x)$ and admit singular expansions of the form*

$$R_{\mathcal{B}'}(z;x) = R_0(x) + R_1(x)(1-z/\rho) + R_2(x)(1-z/\rho)^{3/2} + o\big((1-z/\rho)^{3/2}\big),$$
$$\frac{\partial}{\partial x}R_{\mathcal{B}'}(z;x) = \hat{R}_0(x) + \hat{R}_1(x)(1-z/\rho)^{1/2} + o\big((1-z/\rho)^{1/2}\big),$$

where $\rho(x)$ and the $R_i(x)$ and $\hat{R}_i(x)$ are analytically given. In particular, for $x = SP(\rho_{\mathcal{SP}}) = SP_0^\bullet$ we have

$$\rho \doteq 1.33259, \quad R_0 \doteq 1.46913, \quad R_1 \doteq -2.56404, \quad R_2 \doteq 1.82717,$$
$$\textit{and} \quad \hat{R}_0 \doteq 169.8389, \quad \hat{R}_1 \doteq -621.4279.$$

Proof of Theorem 6.1. As discussed previously, in [4] it was shown that the class $\mathcal{SP}$ is nice. What remains is to check (3.5). Let us abbreviate $SP_0^\bullet = SP^\bullet(\rho_{\mathcal{SP}})$.

Note that the class $\mathcal{B}'$ is isothermic. By applying Lemma 4.1 we readily obtain that

$$I_{\mathcal{B}'}(z) = SP_0^\bullet \cdot \frac{\partial}{\partial x} R_{\mathcal{B}'}(z; SP_0^\bullet) + \frac{SP_0^\bullet\, B''(SP_0^\bullet)}{B'(SP_0^\bullet)} R_{\mathcal{B}'}(z; SP_0^\bullet),$$

where $R_{\mathcal{B}'}(z;x)$ is given in Lemma 6.3. By applying Corollary 6.4 we infer that $I_{\mathcal{B}'}(z)$ has a unique dominant singularity at $\rho = \rho(SP_0^\bullet)$, and that it admits a singular expansion of the form

$$I_{\mathcal{B}'}(z) = I_0 + I_1(1-z/\rho)^{1/2} + o\big((1-z/\rho)^{1/2}\big),$$

where I_0, I_1 are analytically given, and $I_0 \doteq 31.5669$ and $I_1 \doteq -79.5238$. By applying the Transfer Theorem (*e.g.*, Corollary VI.1 in [10]) to the singular expansion of $I_{\mathcal{B}'}(z)$ from Corollary 6.4 for $x = SP_0^\bullet$, we infer that there is a $c > 0$ such that

$$[z^\ell]I_{\mathcal{B}'}(z) = (c + o(1))\mu^\ell \ell^{-3/2}, \quad \text{where} \quad \mu := 1/\rho.$$

Thus, Theorem 3.3 can be applied whenever, say, $k \leqslant \log_{1/\mu} n - 20\log_{1/\mu}\log n$. This gives us that $D_{\mathcal{SP}}(z) = I_{\mathcal{B}'}(z)\cdot e^{\lambda_{\mathcal{SP}}(R_{\mathcal{B}'}(z)-1)}$, where $\lambda_{\mathcal{SP}} = B'(SP_0^\bullet)$. The numerical values claimed in the theorem were calculated with the help of Maple by using the expression for $SP_0^\bullet$ from [4], by solving (6.6a) to determine $D(SP_0^\bullet)$ and by using (6.12) to determine $\lambda_{\mathcal{SP}}$. □

6.1. Proofs

6.1.1. Series-parallel networks

Lemma 6.5 ([19]). *The class $\mathcal{D}$ of SP networks satisfies the equation*

$$\mathcal{D} = e + \mathcal{S} + \mathcal{P}, \tag{6.4}$$

where e is the class of networks consisting of a single edge, and (*cf. Figure 2*)

$$\mathcal{S} = (e+\mathcal{P}) \times \mathcal{Z} \times \mathcal{D} \quad \textit{and} \quad \mathcal{P} = e \times \mathrm{SET}_{\geqslant 1}(\mathcal{S}) + \mathrm{SET}_{\geqslant 2}(\mathcal{S}).$$

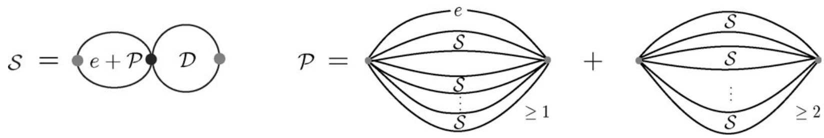

Figure 2. The decomposition of series and parallel networks.

Let $D(x)$, $S(x)$ and $P(x)$ be the egfs of $\mathcal{D}$, $\mathcal{S}$ and $\mathcal{P}$, where x marks the non-pole vertices. From now on we assume that x is a fixed value, and we will always write D for $D(x)$, and analogously S and P. The above lemma then implies

$$D = 1 + S + P, \quad S = (1+P)xD \quad \text{and} \quad P = 2e^S - 2 - S. \tag{6.5}$$

It can be shown [4] that D, S and P satisfy the following relations, which can be derived straightforwardly from the above decomposition, and which we will use several times in our calculations:

$$S = \frac{xD^2}{1+xD} = \log\left(\frac{1+D}{2}\right), \tag{6.6a}$$

$$P + 1 = \frac{D}{1+xD}, \tag{6.6b}$$

$$D' = 2S'e^S = \frac{D^2(1+D)}{1-2xD^2-x^2D^3}. \tag{6.6c}$$

Let D be a graph drawn according to the Boltzmann distribution for $\mathcal{D}$ with parameter x, and denote by $\mathsf{rd}(\mathsf{D})$ the degree of the left pole of D. The following lemma says that D is in a well-defined sense 'symmetric'.

Lemma 6.6. *Let* N *be an element drawn according to the Boltzmann distribution with parameter* x *from one of the classes* $\mathcal{D}$, e, $\mathcal{S}$, $\mathcal{P}$, $e+\mathcal{P}$. *Then the distribution of the degree of the left pole of* N *is the same as the distribution of the degree of the right pole of* N.

Proof. Let $N \in \mathcal{D}$, and let $\mathsf{r}ef(N)$ be the network obtained by fixing any embedding of N and reflecting N at its left pole. Then we have either $\mathsf{r}ef(N) = N$ or $\mathsf{r}ef(\mathsf{r}ef(N)) = N$. Moreover, we clearly have that if $N \in \mathcal{X}$, then also $\mathsf{r}ef(N) \in \mathcal{X}$, where $\mathcal{X}$ is one of $\mathcal{D}$, e, $\mathcal{S}$, $\mathcal{P}$, $e+\mathcal{P}$. This implies that $\mathsf{r}ef$ is a bijection between the elements of $\mathcal{X}$. But then, as $\mathsf{r}ef$ interchanges the degree of the right and of the left pole of all graphs in $\mathcal{X}$, and the probability mass of $N \in \mathcal{X}$ is equal to the probability mass of $\mathsf{r}ef(N)$, the statements follows immediately. □

Below we will write $R_{\mathcal{D}}(z)$ for the probability generating function for $\mathsf{rd}(\mathsf{D})$, *i.e.*, $R_{\mathcal{D}}(z) = \sum_{\ell \geqslant 1} \mathbb{P}[\mathsf{rd}(\mathsf{D}) = \ell] z^\ell$. Similarly, we define the functions $R_{\mathcal{S}}(z)$, $R_{\mathcal{P}}(z)$ and $R_{e+\mathcal{P}}(z)$ for random graphs drawn from $\mathcal{S}$, $\mathcal{P}$, and $e+\mathcal{P}$. The following lemma describes the relations between these four functions.

Lemma 6.7. *$R_{\mathcal{D}}(z)$ satisfies* (6.3). *Moreover,*

$$R_{\mathcal{D}}(z) = R_{e+\mathcal{P}}(z) = R_{\mathcal{S}}(z) \quad \textit{and} \quad R_{\mathcal{P}}(z) = \frac{1}{P}\big((1+P)R_{\mathcal{D}}(z) - z\big). \tag{6.7}$$

Proof. According to the decomposition of $\mathcal{D}$, a random graph from $\mathcal{D}$ is from $\mathcal{S}$ with probability $\frac{S}{D}$, and in this case the probability that its left pole has a specific root degree is given by the corresponding coefficient in $R_{\mathcal{S}}(z)$. Otherwise it is an element of $e + \mathcal{P}$ (with probability $\frac{1+P}{D}$), and the degree of its left pole is given by $R_{e+\mathcal{P}}(z)$. That is,

$$DR_{\mathcal{D}}(z) = SR_{\mathcal{S}}(z) + (1+P)R_{e+\mathcal{P}}(z). \tag{6.8}$$

Similarly, by considering random graphs from $\mathcal{S}$, we obtain

$$SR_{\mathcal{S}}(z) = zxD + xDPR_{\mathcal{P}}(z). \tag{6.9}$$

Furthermore, we have $R_{e+\mathcal{P}}(z) = \frac{1}{1+P}(z + PR_{\mathcal{P}}(z))$, from which by rearranging we obtain

$$PR_{\mathcal{P}}(z) = (1+P)R_{e+\mathcal{P}}(z) - z. \tag{6.10}$$

Then we obtain from (6.8)

$$\begin{aligned} DR_{\mathcal{D}}(z) &\overset{(6.9)}{=} zxD + xDPR_{\mathcal{P}}(z) + (1+P)R_{e+\mathcal{P}}(z) \\ &\overset{(6.10)}{=} zxD + xD((1+P)R_{e+\mathcal{P}}(z) - z) + (1+P)R_{e+\mathcal{P}}(z) \\ &= (1+xD)(1+P)R_{e+\mathcal{P}}(z) \overset{(6.6b)}{=} DR_{e+\mathcal{P}}(z), \end{aligned}$$

which proves the first equality of (6.7). To prove the second equality, we substitute $R_{e+\mathcal{P}}(z) = R_{\mathcal{D}}(z)$ in (6.8), and obtain $SR_{\mathcal{S}}(z) = (D-1-P)R_{\mathcal{D}}(z) = SR_{\mathcal{D}}(z)$, due to (6.5). To prove the last statement in (6.7), we now simply substitute $R_{e+\mathcal{P}}(z) = R_{\mathcal{D}}(z)$ in (6.10).

It remains to show (6.3). Recall that a graph from $\mathcal{P}$ is either an edge merged at its poles with a set consisting of at least one $\mathcal{S}$ network (type I), or a set of $\geqslant 2$ $\mathcal{S}$ networks, merged at their poles (type II). We treat the two cases separately. The Boltzmann probability that a random graph from $\mathcal{P}$ is of type I and consists of exactly i $\mathcal{S}$ networks is $\frac{S^i}{i!P}$. In this case, the probability that the degree of its left pole is exactly ℓ is given by the $(\ell-1)$st coefficient of $R_{\mathcal{S}}(z)^i$. Similarly, the probability that the left pole of a type II graph consisting of i $\mathcal{S}$ networks has degree ℓ is the ℓth coefficient of $R_{\mathcal{S}}(z)^i$. This implies that $R_{\mathcal{P}}(z)$ is related to $R_{\mathcal{S}}(x)$ as follows:

$$R_{\mathcal{P}}(z) = \sum_{\ell \geqslant 0}\left(\sum_{i\geqslant 1}[z^{\ell-1}]R_{\mathcal{S}}(z)^i \frac{S^i}{P}\frac{1}{i!} + \sum_{i \geqslant 2}[z^{\ell}]R_{\mathcal{S}}(z)^i \frac{S^i}{P}\frac{1}{i!}\right)z^{\ell}.$$

This, due to $R_{\mathcal{S}}(z) = R_{\mathcal{D}}(z)$, easily simplifies to

$$R_{\mathcal{P}}(z) = \frac{1}{P}\big((z+1)(e^{SR_{\mathcal{D}}(z)} - 1) - SR_{\mathcal{D}}(z)\big). \tag{6.11}$$

By combining (6.7) and (6.11), we obtain, by applying (6.6a) and (6.6b),

$$(1+P)R_{\mathcal{D}}(z) - z = (z+1)\left(\left(\frac{1+D}{2}\right)^{R_{\mathcal{D}}(z)} - 1\right) - xD(1+P)R_{\mathcal{D}}(z),$$

from which (6.3) follows after rearrangement, together with (6.6b). $\square$

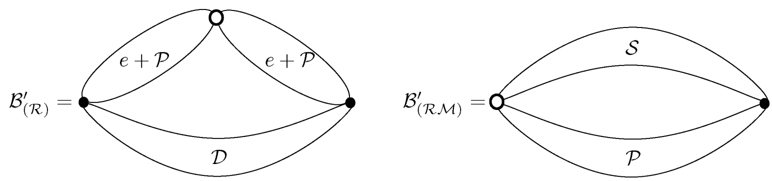

Figure 3. The decomposition of $\mathcal{B}'_{(\mathcal{R})}$ and $\mathcal{B}'_{(\mathcal{RM})}$ (the void vertex denotes the root).

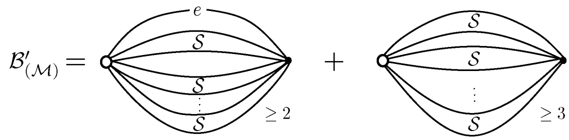

Figure 4. The decomposition of $\mathcal{B}'_{(\mathcal{M})}$ (the void vertex denotes the root).

6.1.2. 2-connected series-parallel graphs. An important tool needed in the proof of Lemma 6.3 is the following fact, which describes how the class $\mathcal{B}'$ can be *directly* decomposed in terms of series-parallel networks.

Lemma 6.8. *The class $\mathcal{B}'$ has the decomposition*

$$\mathcal{B}' = e \times \mathcal{Z} + \mathcal{B}'_{(\mathcal{R})} + \mathcal{B}'_{(\mathcal{M})} - \mathcal{B}'_{(\mathcal{RM})},$$

where e is the class of networks consisting of a single edge, and (cf. Figures 3 and 4)

$$\mathcal{B}'_{(\mathcal{R})} = \mathcal{Z}_2 \times (e+\mathcal{P})^2 \times \mathcal{D}, \quad \mathcal{B}'_{(\mathcal{M})} = \mathcal{Z} \times \big(e \times \textsc{Set}_{\geqslant 2}(\mathcal{S}) + \textsc{Set}_{\geqslant 3}(\mathcal{S})\big),$$
$$\mathcal{B}'_{(\mathcal{RM})} = \mathcal{Z} \times \mathcal{P} \times \mathcal{S},$$

where $\mathcal{Z}_2$ is the class consisting of a single graph with two labelled isolated vertices.

Proof. The proof follows directly from the main result in [6], and the fact that there are no 3-connected series-parallel graphs. □

From the above decomposition we can immediately derive the following relations for the generating functions, which we can simplify and express in terms of D by exploiting (6.6a) and (6.6b):

$$B'_{(\mathcal{R})}(x) = \frac{x^2}{2}(P+1)^2 D = \frac{D^3x^2}{2(1+xD)^2},$$
$$B'_{(\mathcal{M})}(x) = x(e^S - 1 - S) + x\left(e^S - 1 - \frac{S^2}{2} - S\right)$$
$$= x\left(D - 2\frac{xD^2}{1+xD} - 1 - \frac{1}{2}\frac{x^2D^4}{(1+xD)^2}\right),$$
$$B'_{(\mathcal{RM})}(x) = x^2(1+P)DP = \frac{x^2D^2(D-1-xD)}{(1+xD)^2}.$$

With this we can reproduce the result from [4], where $B'(x)$ was determined by using (6.1):

$$B'(x) = x + B'_{(\mathcal{R})}(x) + B'_{(\mathcal{M})}(x) - B'_{(\mathcal{RM})}(x) = -\frac{xD(xD^2-2)}{2(1+xD)}. \tag{6.12}$$

Proof of Lemma 6.3. First, note that the second statement of the lemma was already proved in Lemma 6.7. The remaining proof uses ideas similar to the proof of Lemma 6.7; we shall therefore concentrate on the main steps. From the decomposition of $\mathcal{B}'$ and the auxiliary classes $\mathcal{B}'_{(\mathcal{R})}$, $\mathcal{B}'_{(\mathcal{M})}$ and $\mathcal{B}'_{(\mathcal{RM})}$ (Figures 3 and 4), we obtain

$$\begin{aligned} R_{\mathcal{B}'_{(\mathcal{R})}}(z) &= R_{e+\mathcal{P}}(z)^2 \stackrel{(6.7)}{=} R_{\mathcal{D}}(z)^2, \\ R_{\mathcal{B}'_{(\mathcal{RM})}}(z) &= R_{\mathcal{P}}(z)R_{\mathcal{S}}(z) \stackrel{(6.7)}{=} R_{\mathcal{P}}(z)R_{\mathcal{D}}(z). \end{aligned} \tag{6.13}$$

From (6.7) it follows that $R_{\mathcal{P}}(z) = \frac{1+P}{P}R_{\mathcal{D}}(z) - \frac{z}{P}$, and therefore we can express $R_{\mathcal{B}'_{(\mathcal{RM})}}(z)$ as a function of only $R_{\mathcal{D}}(z)$ and D, namely

$$R_{\mathcal{B}'_{(\mathcal{RM})}}(z) = \frac{1+P}{P}R_{\mathcal{D}}(z)^2 - \frac{z}{P}R_{\mathcal{D}}(z) \stackrel{(6.6)}{=} \frac{R_{\mathcal{D}}(z)((1+xD)z - DR_{\mathcal{D}}(z))}{1+xD-D}.$$

Furthermore,

$$\begin{aligned} [z^\ell]R_{\mathcal{B}'_{(\mathcal{M})}}(z) &= \sum_{i\geqslant 2}[z^{\ell-1}]R_{\mathcal{S}}(z)^i \frac{1}{B'_{(\mathcal{M})}}\frac{S^i}{i!}x + \sum_{i\geqslant 3}[z^\ell]R_{\mathcal{S}}(z)^i\frac{1}{B'_{(\mathcal{M})}}\frac{S^i}{i!}x \\ &= \frac{x}{B'_{(\mathcal{M})}}\left(\sum_{i\geqslant 2}[z^\ell]z\frac{(SR_{\mathcal{S}}(z))^i}{i!} + \sum_{i\geqslant 3}[z^\ell]\frac{(SR_{\mathcal{S}}(z))^i}{i!}\right) \\ &= \frac{x}{B'_{(\mathcal{M})}}[z^\ell]\left((z+1)(e^{SR_{\mathcal{D}}(z)} - 1 - SR_{\mathcal{D}}(z)) - \frac{(SR_{\mathcal{D}}(z))^2}{2}\right). \end{aligned} \tag{6.14}$$

We obtain an expression for $R_{\mathcal{B}'_{(\mathcal{M})}}(z)$ by summing up the above term for all ℓ. This can be written in terms of only $R_{\mathcal{D}}(z)$ and D as follows. Note that $(z+1)e^{SR_{\mathcal{D}}(z)} = DR_{\mathcal{D}}(z)+1$, due to (6.3) and (6.6a). Then we obtain

$$\begin{aligned} R_{\mathcal{B}'_{(\mathcal{M})}}(z) &= \frac{x}{B'_{(\mathcal{M})}}\left((z+1)e^{SR_{\mathcal{D}}(z)} - (z+1)(1+SR_{\mathcal{D}}(z)) - \frac{(SR_{\mathcal{D}}(z))^2}{2}\right) \\ &= \frac{x}{B'_{(\mathcal{M})}}\left(DR_{\mathcal{D}}(z) - z - (z+1)\frac{xD^2}{1+xD}R_{\mathcal{D}}(z) - \frac{x^2D^4}{2(1+xD)^2}R_{\mathcal{D}}(z)^2\right). \end{aligned}$$

Putting everything together, we obtain

$$R_{\mathcal{B}'}(z) = \frac{1}{B'}\big(xz + B'_{(\mathcal{R})}R_{B'_{(\mathcal{R})}}(z) + B'_{(\mathcal{M})}R_{B'_{(\mathcal{M})}}(z) - B'_{(\mathcal{RM})}R_{B'_{(\mathcal{RM})}}(z)\big), \tag{6.15}$$

which simplifies to the form stated in the lemma. □

6.2. Remaining proofs

Proof of Corollary 6.4. Solving (6.3) yields the explicit form (as usual we write D for $D(x)$, and recall identity (6.6a)):

$$R_{\mathcal{D}}(z;x) = T\left(\frac{xDe^{-\frac{xD}{1+xD}}}{1+xD}(z+1)\right)\frac{1+xD}{xD^2} - \frac{1}{D} =: T(\alpha(z+1))\beta+\gamma \tag{6.16}$$

for $R_{\mathcal{D}}(z;x)$, with $T(z)$ the *tree function*, which satisfies $T(z) = ze^{T(z)}$. $T(z)$ has a unique dominant singularity at $z = e^{-1}$, and therefore the singularity $\rho(x)$ of $R_{\mathcal{D}}(z;x)$ is obtained by setting $\alpha(z+1)$ equal to e^{-1}. By solving this equation we obtain

$$\rho = \rho(x) = e^{-\frac{1}{1+xD}}\left(1+\frac{1}{xD}\right) - 1.$$

It is also well known (see, *e.g.*, [10]) that the singular expansion of the tree function is

$$T(z) = 1 + t_1(1-ez)^{1/2} + t_2(1-ez) + t_3(1-ez)^{3/2} + o\big((1-ez)^2\big),$$

where

$$t_1 = -\sqrt{2}, \quad t_2 = \frac{2}{3} \quad \text{and} \quad t_3 = -\frac{11}{36}\sqrt{2}.$$

Moreover, an easy calculation shows that

$$T'(z) = \frac{T(z)}{z(1-T(z))},$$

which implies

$$T'(z) = t'_{-1}(1-ez)^{-1/2} + t'_0 + t'_1(1-ez)^{1/2} + o(1-ez), \tag{6.17}$$

where

$$t'_{-1} = \frac{e}{\sqrt{2}}, \quad t'_0 = -\frac{2e}{3} \quad \text{and} \quad t'_1 = \frac{11\sqrt{2}e}{24}.$$

We note that, due to $1 - e\alpha(z+1) = (1-e\alpha)(1-z/\rho)$, we have that $\alpha(z+1) \to \frac{1}{e}$ is equivalent to $z \to \rho$. If we therefore substitute z with $\alpha(z+1)$ in the expansion above, we obtain the singular expansion for $T(\alpha(z+1))$:

$$\begin{aligned} T(\alpha(z+1)) &= 1 + t_1\big(1-e\alpha(z+1)\big)^{1/2} + t_2\big(1-e\alpha(z+1)\big) + t_3\big(1-e\alpha(z+1)\big)^{3/2} + \cdots \\ &=: \tilde{r}_0 + \tilde{r}_1(1-z/\rho)^{1/2} + \tilde{r}_2(1-z/\rho) + \tilde{r}_3(1-z/\rho)^{3/2} + o\big((1-z/\rho)^2\big), \end{aligned}$$

where $\tilde{r}_i = t_i(1-e\alpha)^{i/2}$, for $0 \leqslant i \leqslant 3$. Given this, and knowing that $R_{\mathcal{D}}(z) = T(\alpha(z+1))\beta+\gamma$, the singular expansion

$$R_{\mathcal{D}}(z) = r_0 + r_1(1-z/\rho)^{1/2} + r_2(1-z/\rho) + r_3(1-z/\rho)^{3/2} + o\big((1-z/\rho)^2\big)$$

of $R_{\mathcal{D}}(z;x)$ is easy to calculate, with $r_0 = \tilde{r}_0 \cdot \beta + \gamma$ and $r_i = \tilde{r}_i\beta$ for $1 \leqslant i \leqslant 3$. Finally, to compute the singular expansion of $R_{\mathcal{B}'}(z;x)$ we recall (6.2). Notice that $R_{\mathcal{B}'}(z;x)$ and $R_{\mathcal{D}}(z;x)$ have the same singularity $\rho(x)$. Then, by plugging into (6.2) the singular expansion of $R_{\mathcal{D}}(z;x)$ derived above, we readily obtain the expansion of $R_{\mathcal{B}'}(z;x)$. To see why the term $(1-z/\rho)^{1/2}$ is missing, observe that $2r_0r_1xD^2 - 2r_1 = 0$.

Next we derive the singular expansion of $\frac{\partial}{\partial x}R_{\mathcal{D}}(z;x)$. By taking the derivative of the expression (6.16), we obtain

$$\frac{\partial}{\partial x}R_{\mathcal{D}}(z;x) = T'(\alpha(z+1))(z+1)\alpha'\beta + T(\alpha(z+1))\beta' + \gamma'.$$

Together with (6.17) and the fact that

$$z+1 = \frac{1-(1-e\alpha)(1-z/\rho)}{e\alpha},$$

this yields

$$\frac{\partial}{\partial x}R_{\mathcal{D}}(z;x) = r'_{-1}(1-z/\rho)^{-1/2} + r'_0 + r'_1(1-z/\rho)^{1/2} + o\big((1-z/\rho)^{1/2}\big),$$

where

$$r'_{-1} = \frac{t_{-1}(1-e\alpha)^{-1/2}\alpha'\beta}{e\alpha}, \quad r'_0 = \frac{t'_0\alpha'\beta}{e\alpha} + \beta'\tilde{r}_0 + \gamma'$$
$$\text{and} \quad r'_1 = \frac{(t'_1 - t'_{-1})(1-e\alpha)^{1/2}\alpha'\beta}{e\alpha} + \beta'\tilde{r}_1.$$

Finally, to compute the singular expansion of $\frac{\partial}{\partial x}R_{\mathcal{B}'}(z;x)$ we take the derivative of the right-hand side of (6.2) with respect to x, and use the singular expansion of $\frac{\partial}{\partial x}R_{\mathcal{D}}(z;x)$. An easy (but lengthy) computation shows also in this case that the coefficient of the term $(1-z/\rho)^{-1/2}$ vanishes in the expansion of $\frac{\partial}{\partial x}R_{\mathcal{B}'}(z;x)$.

The approximate numerical values in the statement of the corollary were obtained with the help of MAPLE, where we plugged into the above equations the value $x = SP_0^\bullet$ from [4], and we obtained $D(SP_0^\bullet)$ by solving (6.6a) numerically for D. □

Appendix

Proof of Lemma 2.4. Let

$$f(z) := e^{\frac{\alpha z+\beta z^2}{1-z}} \cdot \frac{1}{(1-z)^\gamma}.$$

By the Cauchy integral formula we have

$$[z^n]f(z) = \frac{1}{2i\pi}\oint_{\mathcal{C}} \frac{f(z)}{z^{n+1}}dz = \frac{1}{2i\pi}\oint_{\mathcal{C}} e^{\log f(z)-(n+1)\log z}dz =: \frac{1}{2i\pi}\oint_{\mathcal{C}} e^{h(z)}dz, \tag{A.1}$$

where $\mathcal{C}$ is any contour enclosing the origin, and lying completely in the domain of f. To estimate the integral we will use the *saddle point method*, which is commonly used when one wants to determine the asymptotic behaviour of integrals that involve a large parameter, and are simultaneously subject to huge variations. For an excellent overview and numerous applications we refer the reader to [10].

The main idea of the saddle point method is to choose $\mathcal{C}$ such that the integrand has a unique maximum on $\mathcal{C}$, and the main contribution to the integral comes from a small neighbourhood of this maximum. Let $r = r(n) := 1 - \sqrt{(\alpha+\beta)/n}$, and choose $\mathcal{C} := \{re^{i\theta} \mid \theta \in (-\pi,\pi]\}$, *i.e.*, $\mathcal{C}$ is the cycle with radius r with centre in the origin. (The choice of r might seem at this point somewhat arbitrary. However, r is an approximate solution of $h'(z) = 0$, and hence the actual (global) maximum of the function is located very near to r. As we shall see later, this approximation will suffice for our purpose.)

Our aim is now to 'split' $\mathcal{C}$ into two disjoint parts $\mathcal{C}_0$ and $\mathcal{C}_1$ such that the contribution of $\oint_{\mathcal{C}_0}$ to $\oint_{\mathcal{C}}$ is negligible, and $\oint_{\mathcal{C}_1}$ is so well behaved, that we can approximate it quite precisely. For this, let $\theta_0 = \theta_0(n) := n^{-17/24}$, and let $\mathcal{C}_1 := \{re^{i\theta} \mid |\theta| < \theta_0\}$ and $\mathcal{C}_0 := \mathcal{C} \setminus \mathcal{C}_1$.

First we show an upper bound for $\oint_{\mathcal{C}_0} \frac{f(z)}{z^{n+1}}dz$. For $|\theta| \geqslant \theta_0$ we have

$$\begin{aligned}|e^{\frac{\alpha z+\beta z^2}{1-z}}|_{z=re^{i\theta}} &= |e^{\Re(\frac{\alpha z+\beta z^2}{1-z})}|_{z=re^{i\theta}}\\ &= \exp\left(\frac{(2\beta r\cos(\theta)^2 + \alpha\cos(\theta) - r^2\beta\cos(\theta) - r\beta - r\alpha)r}{1 - 2r\cos(\theta) + r^2}\right).\end{aligned}$$

Denote the exponent in the above expression by $q = q(\theta)$, and abbreviate $c = \cos(\theta)$. By differentiating q we see that it becomes maximal either when $\sin(\theta) = 0$, *i.e.*, $\theta = \pi$ (as $|\theta| \geqslant \theta_0$), or when

$$4\beta rc - 4\beta r^2c^2 + 4\beta r^3c + \alpha - \alpha r^2 - 3\beta r^2 - \beta r^4 = 0 \leftrightarrow c = \frac{\beta r^2 + \beta \pm \sqrt{\beta(\beta+\alpha)(1-r^2)}}{2\beta r}.$$

Denote the above two values of c by c_+ and c_-. Then a straightforward calculation shows that

$$q(\pi) = \frac{r(r\beta - \alpha)}{1+r} \quad \text{and} \quad q(\arccos(c_\pm)) = -\frac{1}{2}(r^2\beta + 2\beta + \alpha) \mp \sqrt{\beta(\beta+\alpha)(1-r^2)},$$

which are all $\Theta(1)$. Hence $|e^{\frac{\alpha z+\beta z^2}{1-z}}|$ becomes maximal at $\theta = \theta_0$, and with the estimate

$$\cos(\theta_0) = 1 - \frac{n^{-17/12}}{2} + \Theta(n^{-17/6})$$

we obtain

$$|e^{\frac{\alpha z+\beta z^2}{1-z}}|_{z=re^{i\theta}} \leqslant e^{q(\theta_0)} \leqslant e^{\sqrt{(\alpha+\beta)n} - \Omega(n^{1/12})}, \quad \theta \in (-\pi, \pi] \setminus [-\theta_0, \theta_0].$$

With this, the integral over $\mathcal{C}_0$ is at most

$$\begin{aligned}\oint_{\mathcal{C}_0} \frac{f(z)}{z^{n+1}}dz &\leqslant \|\mathcal{C}\| \cdot \max_{z\in\mathcal{C}_0}|f(z)| \leqslant \Theta(1) \cdot \max_{z\in\mathcal{C}_0}|e^{\frac{\alpha z+\beta z^2}{1-z}}| \cdot |(1-z)^{-\gamma}| \cdot |z^{-n-1}|\\ &\leqslant e^{\sqrt{(\alpha+\beta)n} - \Omega(n^{1/12})} \cdot n^{\Theta(1)} \cdot \left(1 - \sqrt{\frac{\alpha+\beta}{n}}\right)^{-n-1} = e^{2\sqrt{(\alpha+\beta)n} - \Omega(n^{1/12})}. \qquad \text{(A.2)}\end{aligned}$$

In words, $\oint_{\mathcal{C}_0}$ is exponentially smaller than $e^{2\sqrt{(\alpha+\beta)n}}$. Next we determine the asymptotic value of $\oint_{\mathcal{C}_1} \frac{f(z)}{z^{n+1}}dz$. Let us collect some basic properties of h. Note that

$$\begin{aligned}h(r) &= 2\sqrt{(\alpha+\beta)n} + \frac{1}{2}\gamma\log n - \frac{1}{2}(3\beta + \alpha + \gamma\log(\alpha+\beta)) + o(1), \quad \text{and}\\ h'(r) &= \frac{\gamma - \alpha - \beta}{\sqrt{\alpha+\beta}}\sqrt{n} + \Theta(1), \quad h''(r) = \frac{2}{\sqrt{\alpha+\beta}}n^{3/2} + \Theta(n).\end{aligned}$$

Moreover, note that $|h'''(z)| = \mathcal{O}(n^2)$, as h''' is proportional to $(1-z)^{-4}$ for $z \in \mathcal{C}$. For the remainder, let $|\theta| \leqslant \theta_0$. Since for $z \in \mathcal{C}_1$ we have $z - r = r(e^{i\theta} - 1) = ir\theta + \Theta(\theta^2)$, we may approximate h with its Taylor series,

$$\begin{aligned} h(z) &= h(r) + h'(r)(z-r) + \frac{1}{2}h''(r)(z-r)^2 + \Theta\Big(\max_{z\in\mathcal{C}_1} h'''(z)(z-r)^3\Big) \\ &\overset{(z=re^{i\theta})}{=} h(r) - n^{3/2}\frac{r^2\theta^2}{\sqrt{\alpha+\beta}} + o(1+i). \end{aligned}$$

Write $x \sim y$ if $x = (1+o(1))y$ for $n \to \infty$. With the above approximation, we obtain

$$\begin{aligned} \oint_{\mathcal{C}_1} \frac{f(z)}{z^{n+1}} dz &\sim e^{h(r)} \cdot i \cdot \int_{-\theta_0}^{\theta_0} e^{-n^{3/2}\frac{r^2\theta^2}{\sqrt{\alpha+\beta}}} d\theta \\ &= e^{h(r)} \cdot i \cdot \int_{-n^{1/24}}^{n^{1/24}} e^{-\frac{r^2}{\sqrt{\alpha+\beta}}x^2} n^{-3/4} dx \sim ie^{h(r)} n^{-3/4} \pi^{1/2} (\alpha+\beta)^{1/4}. \end{aligned}$$

Hence, a simple calculation shows that $\frac{1}{2i\pi}\oint_{\mathcal{C}1}$ is, up to polynomial factors, equal to $e^{2\sqrt{(\alpha+\beta)n}}$, *i.e.*, due to (A.2), it is asymptotically much larger than $\oint_{\mathcal{C}0}$. Finally, by plugging in the precise value of $h(r)$, we immediately obtain (2.3). $\square$

Proof of Lemma 3.6. Let $\delta \geqslant 0$ be such that $x + \delta \leqslant \frac{x+\rho_\mathcal{G}}{2}$. Then, as $G(z)$ has only non-negative coefficients and is analytic in its disc of convergence, there is an absolute constant $c > 0$ such that

$$G(x+\delta) \leqslant G(x) + \delta G'(x) + \delta^2 c.$$

Here, one might, for example, choose $c = \frac{1}{2}G''(\frac{x+\rho_\mathcal{G}}{2})$. A straightforward induction argument over N shows that, for any t,

$$\mathbb{P}\left[\sum_{i=1}^{N} |\mathsf{G}_i| = t\right] = \frac{x^t [z^t] G(z)^N}{G(x)^N}.$$

Note that as $G(z)$ has only non-negative coefficients, then for any $0 < r < \rho_\mathcal{G}$ we have that $[z^n]G(z)^N \leqslant G(r)^N r^{-n}$. Let $s = (1+\varepsilon)vN$. Using the above facts, we obtain for any $x < r \leqslant (x+\rho_\mathcal{G})/2$

$$p_s := \mathbb{P}\left[\sum_{i=1}^{N} |\mathsf{G}_i| \geqslant s\right] \leqslant \left(\frac{x}{r}\right)^s \left(\frac{G(r)}{G(x)}\right)^N \frac{r}{r-x}. \tag{A 3}$$

Write $r = x + \delta$. Then we may estimate

$$\left(\frac{x}{r}\right)^s \leqslant \exp\left\{-\frac{\delta s}{x+\delta}\right\} \leqslant \exp\left\{-\frac{\delta s}{x} + \frac{\delta^2 s}{x^2}\right\}.$$

Moreover, by exploiting the Taylor expansion of G around x, we obtain

$$\left(\frac{G(r)}{G(x)}\right)^N \leqslant \left(1 + \delta\frac{G'(x)}{G(x)} + \delta^2\frac{c}{G(x)}\right)^N \leqslant \exp\left\{\left(\delta\frac{G'(x)}{G(x)} + \delta^2\frac{c}{G(x)}\right)N\right\}.$$

Recalling that $v = \frac{xG'(x)}{G(x)}$, we can combine the above estimates and deduce from (A 3) that

$$\log p_s \leqslant \delta \frac{G'(x)}{G(x)} \left(-\varepsilon + \delta \left(\frac{1+\varepsilon}{x} + \frac{c}{G'(x)} \right) \right) N + \log(r/(r-x)).$$

All the above bounds are true for any $\delta \geqslant 0$ such that $x + \delta \leqslant (x + \rho_{\mathcal{G}})/2$. For the remaining calculation we set

$$\delta = \min\left\{ \frac{\varepsilon}{2(\frac{1+\varepsilon}{x} + \frac{c}{G'(x)})}, \frac{x+\rho_{\mathcal{G}}}{2} - x \right\}.$$

Observe that this implies that there exist constants $C_1 = C_1(x)$ and $C_2 = C_2(x)$ such that $\delta \geqslant C_1 \varepsilon x/(1+\varepsilon)$ and $\log(r/(r-x)) = \log(1 + x/\delta) \leqslant C_2(1 + 1/\varepsilon)$. Hence, we obtain

$$\log p_s \leqslant \; -\delta \frac{G'(x)}{G(x)} \frac{\varepsilon}{2} N + \log(r/(r-x)) \leqslant -C_1 \frac{\varepsilon^2}{1+\varepsilon} \frac{xG'(x)}{G(x)} N + C_2(1 + 1/\varepsilon),$$

and the proof is complete. □

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